Infinitesimal Models of Algebraic Theories

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I would like to dedicate this thesis to my mother, who has always supported me unconditionally in my pursuit of knowledge and truth.

Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the acknowledgements and specified in the text. It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge, or any other university or similar institution except as declared in the acknowledgements and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma, or other qualification at the University of Cambridge, or any other university or similar institution except as declared in the acknowledgements and specified in the text.

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Abstract

Smooth manifolds have been always understood intuitively as spaces that are infinitesimally linear at each point, and thus infinitesimally affine when forgetting about the base point. The aim of this thesis is to develop a general theory of infinitesimal models of algebraic theories that provides us with a formalisation of these notions, and which is in accordance with the intuition when applied in the context of Synthetic Differential Geometry. This allows us to study well-known geometric structures and concepts from the viewpoint of infinitesimal geometric algebra.

Infinitesimal models of algebraic theories generalise the notion of a model by allowing the operations of the theory to be interpreted as partial operations rather than total operations. The structures specifying the domains of definition are the infinitesimal structures. We study and compare two definitions of infinitesimal models: actions of a clone on infinitesimal structures and models of the infinitesimalisation of an algebraic theory in cartesian logic. The last construction can be extended to first-order theories, which allows us to define infinitesimally euclidean and projective spaces, in principle.

As regards the category of infinitesimal models of an algebraic theory in a Grothendieck topos we prove that it is regular and locally presentable. Taking a Grothendieck topos as a base we study lifts of colimits along the forgetful functor with a focus on the properties of the category of infinitesimally affine spaces.

We conclude with applications to Synthetic Differential Geometry. Firstly, with the help of syntactic categories we show that the formal dual of every smooth ring is an infinitesimally affine space with respect to an infinitesimal structure based on nil-square infinitesimals. This gives us a good supply of infinitesimally affine spaces in every well-adapted model of Synthetic Differential Geometry. In particular, it shows that every smooth manifold is infinitesimally affine and that every smooth map preserves this structure. In the second application we develop some basic theory of smooth loci and formal manifolds in naive Synthetic Differential Geometry using infinitesimal geometric algebra.

Contents

Introduction

Smooth manifolds have been always understood intuitively as spaces that become linear spaces in the infinitesimal neighbourhood of each point. The infinitesimal geometry underlying a manifold is thus affine geometry.

To be able to make this last statement precise we need a notion of what it means for points of a manifold to be infinitesimally close to each other, first. We follow Kock in [Koc06] and say that two points $P, Q \in M$ of a (formal) manifold are 1-neighbours, if $(P-Q, P-Q)$ is annihilated by any bilinear form when everything is considered in a chart. This yields a symmetric, reflexive relation '$\sim_1$' on M, which is called the first neighbourhood of the diagonal. An n-tuple of points $(P_1, \ldots, P_n) \in M^n$ is an infinitesimal neighbourhood (an infinitesimal simplex in the terminology of Kock), if all points are mutual 1-neighbours. In [Koc09] Kock has further shown that we can form affine combinations of the points of an infinitesimal neighbourhood, and that the resulting point is a neighbour of each point. The question that has been left unanswered is thus: *In what sense is a formal manifold a model of the theory of affine combinations?* This is the initial question that has lead to the investigations resulting in this thesis.

It is clear that the affine operations are only defined partially. The domains of definition are given by the spaces of infinitesimal neighbourhoods $M\langle n\rangle$, where n stands for the number of points in a neighbourhood. The n-ary operations of affine combinations are only defined on $M\langle n\rangle$ instead of M^n. The structure that we extract from this example is called an *infinitesimal structure*. It satisfies $M\langle 1\rangle = M$, but $M\langle 0\rangle$ is terminal and not an initial object, as we would intuitively choose for affine combinations. (The necessity for this choice becomes clear once we attempt to define infinitesimal models of an algebraic theory with constants.) Furthermore, for a formal manifold the infinitesimal structure $M\langle -\rangle$ is generated by $M\langle 2\rangle$, which is the relation '$\sim_1$'. We do not require general infinitesimal structures to have this property. This gives us more flexibility for the gluing of infinitesimal structures.

The next problem we face is to find a suitable representation of the theory of affine combinations; we are dealing with the theory of affine combinations over a non-trivial ring R in some Grothendieck topos $\mathcal{S}$, after all. A natural choice is to represent the theory of affine combinations over R by a *clone* $\mathcal{A}_R$ in $\mathcal{S}$. In the internal language of $\mathcal{S}$ each $\mathcal{A}_R(n)$ is the subspace of $\vec{\lambda} \in R^n$ such that the coefficients sum up to 1. We can show that the syntactic category of the algebraic theory of rings has a clone of affine combinations of the generic ring. Every ring in a finite-limit category thus comes along with a clone of affine combinations over that ring. Also, forming affine combinations of an infinitesimal neighbourhood amounts to an action of $\mathcal{A}_R(n)$ on $M\langle n\rangle$.

It remains to clarify what we mean by an action of a clone on an infinitesimal structure. For the most part we can simply copy the definition of the action of a clone on a set. To make associativity work, however, we have to introduce a new axiom: the *neighbourhood axiom*. For the affine combinations in M it says that if we fix an infinitesimal neighbourhood $(P_1, \ldots, P_n)$ then all affine combinations of theses points will be mutual 1-neighbours. In this way it is ensured that we can substitute affine combinations into each other as we are used to. The neighbourhood axiom is a slightly stronger requirement than what has been proved by Kock; but it is not difficult to see that M satisfies it as well.

From the consideration of the formal manifold M we have finally arrived at a structural definition of an infinitesimal model of the theory of affine combinations: it is the action of the clone of affine combinations on an infinitesimal structure satisfying the neighbourhood axiom. Since clones (in the category Set) are equivalent to algebraic theories, we obtain a notion of *infinitesimal models of algebraic theories*.

For the sake of applications to SDG we would like to have a notion of infinitesimal model for the syntactic presentation of theories, also. Given a (one-sorted) algebraic theory $\mathbb{T}$ we have to construct its *infinitesimalisation* $\mathrm{I}[\mathbb{T}]$, which amounts to introduce a generic infinitesimal structure and make the operations of $\mathbb{T}$ partial. In the framework of categorical first-order logic both, the definition of $\mathrm{I}[\mathbb{T}]$ and proving it equivalent to the clone approach for clones in Set are rather cumbersome. This is because the formalism is not designed to deal with partial operations conveniently.

It is possible to extend the infinitesimalisation construction to first-order theories. We give a definition of infinitesimalisation of cartesian theories with the aim to apply it to the many-sorted algebraic theory of clone algebras and to the essentially algebraic theory of affine spaces. However, for both examples it becomes clear that the construction of infinitesimalisation we define is not refined enough. At this stage it can merely serve as a guide for how to do such a construction.

Motivation and further research

The motivation behind the research conducted in this thesis is two-fold. First of all, having a notion of infinitesimal model of algebraic theories allows us to define new types of geometric space in *Synthetic Differential Geometry* (SDG), namely *infinitesimally affine spaces*, *infinitesimal groups* and *infinitesimal vector spaces*. We can develop an infinitesimal geometric algebra, which can then be applied to study structures, concepts and problems in SDG. That such applications are potentially rich is already indicated by the many successful applications of infinitesimal affine combinations to SDG in [Koc09]. We shall give a basic account of infinitesimal geometric algebra in chapter 3.3. Unfortunately, we are not able to include more sophisticated results, like that infinitesimal affine connections on neighbourhood retracts are equivalent to infinitesimal affine structures on the infinitesimal structure of second-order neighbours.

The second motivation draws from the pursuit of the following problems. We would like to define and understand spaces, which are infinitesimally euclidean, infinitesimally projective, and, more generally, infinitesimal models of other types of homogeneous geometries. Once we have such notions we would like to understand their relationship with Cartan geometries.

We also have the hierarchy projective→affine→(euclidean, elliptic, hyperbolic). Affine geometry can be obtained from projective geometry by singling out a hyperplane at infinity. The other three geometries are obtained by introducing a quadric in addition to the hyperplane at infinity. *Can we reproduce this hierarchy for the infinitesimal models of these geometries?*

In this thesis we will only get as far as showing that we can define infinitesimal models for first-order theories, which allows us to define infinitesimally projective and euclidean spaces, in principle. However, as we have remarked before, the construction of infinitesimalisation of first-order theories, which makes the definition of infinitesimal model possible, is still very unrefined at this stage.

Structure and outline of the thesis

The thesis is split into three chapters. At the beginning of each chapter we include an outline of the content and state the main results.

The first chapter The aim of the first chapter is to introduce and compare three viewpoints on algebraic theories: the syntactic approach via categorical logic, (enriched) Lawvere theories and clones and their algebras. We introduce algebraic theories in the formalism of cartesian logic first. Since we will be working with cartesian logic a lot in this thesis we give a detailed presentation and a proof of the soundness theorem. To keep the exposition at a reasonable length we have not included the clauses for coherent and geometric logic, even though we will use these fragments at the end of the second and in the third chapter.

The section about enriched Lawvere theories is kept very brief. This is because, when compared to clones and the syntactic approach, Lawvere theories play no further role for the theory of infinitesimal models as developed in this thesis. On the other hand, we develop the theory of clones in much more detail. One reason is that clones in finite-product categories seem to not have been considered in the literature yet; the other is to compare possible definitions of clone algebras, which will give us more flexibility in applying the theory later on. With the comparison theorems in mind we show that any algebraic theory induces a clone in the category Set such that the respective categories of models are isomorphic. The most important example of a clone for us, however, is the clone of affine combinations. We briefly review the notion of syntactic category and define the clone of affine combinations in the syntactic category of the theory $\mathbb{T}$ of (commutative, unital) K-algebras for a commutative ring K.

In the last section we prove the comparison theorem that establishes the equivalence of Set-clones with the syntactic approach. To be able to compare the syntactic approach with clones in finite-limit categories we introduce the many-sorted algebraic theory of clones and their algebras. The infinitesimalisation of this theory will be important for the syntactic description of clones and their infinitesimal algebras later on. We also prove the equivalence of the category of Lawvere theories enriched in a complete and cocomplete cartesian closed category $\mathcal{V}$ and the category of clones in $\mathcal{V}$. We have included this result only to show that, by extending the well-known result from Set, clones may be understood as enriched algebraic theories.

The second chapter The second chapter on infinitesimal models of algebraic theories is the heart of this thesis. It has three aims. The first aim is to introduce and compare two viewpoints on infinitesimal models of algebraic theories based on clones and the syntactic approach. The second aim is to prove that every object in the syntactic category of the theory $\mathbb{T}$ of K-algebras is an infinitesimal model of the clone of affine combinations, and that all morphisms preserve this structure. The third aim is to study the properties of the categories of infinitesimal models of an algebraic theory.

We begin by introducing the notion of infinitesimal structure and show that every object in the syntactic category of the theory $\mathbb{T}$ of K-algebras admits a nil-square infinitesimal structure, which is the infinitesimal structure generated by the first neighbourhood of the diagonal as in SDG. If we restrict ourselves to the isomorphism dense subcategory generated by Horn formulae-in-context, then, with the help of a polynomial version of Hadamard's lemma, we can show that every morphism preserves this infinitesimal structure. Building on results from linear algebra over commutative rings and computational commutative algebra we are able to extend this result to a certain classes of formulae-in-context containing the cartesian existential quantifier. With the notion of infinitesimal structure at hand we can define infinitesimal models of algebraic theories as actions of clones on infinitesimal structures, which we call infinitesimal algebras. We generalise the observations made in the first chapter to this setting. The infinitesimal structure on a Horn formula-in-context is closed under affine combinations making it an infinitesimal model of the clone of affine combinations. We use the isomorphism of the full subcategory generated by the Horn formulae-in-context and the opposite category of finitely presented K-algebras to show that every morphism between Horn formulae-in-context preserves this action. This argument generalises easily to the opposite category of all K-algebras.

Regarding a notion of an infinitesimal model of an algebraic theory from the syntactic viewpoint, we introduce the construction of infinitesimalisation of algebraic theories. This construction takes an algebraic theory, replaces the sort with an infinitesimal structure, redefines the original operations as partial operations and modifies the axioms accordingly. The resulting theory is a cartesian theory and we prove a comparison theorem stating that the category of models of the infinitesimalisation of an algebraic theory is isomorphic to the category of infinitesimal algebras of the corresponding Set-clone. To be able to capture infinitesimal algebras of all clones syntactically, and, in particular, infinitesimal models of the theory of affine combinations, we need to extend the infinitesimalisation to the class of cartesian theories. The construction we propose here turns out to be too general. We use it as a guide only and give the infinitesimalisations of the many sorted algebraic theory of clones and their algebras as well as of the essentially algebraic theory of affine spaces explicitly. The comparison theorems for both confirm that our constructions are correct.

We study the properties of categories of infinitesimal models over a Grothendieck topos in the last section. The restriction to Grothendieck toposes is due to our interest in applications to SDG. It is for the same reason that we are mostly interested in lifting properties of the forgetful functor from the models to the base topos. Using established results we can show easily that the category of infinitesimal models is locally presentable and that the forgetful functor lifts limits and filtered colimits uniquely. Regarding lifting properties for colimits the key result is that the forgetful functor lifts pushouts of morphisms, which reflect the infinitesimal structure. It is this property, in general, which allows us to define the infinitesimal model on the colimit by taking the joins of the images of the respective structures under the morphisms of the colimit cone. Since the initial object is given by the (total) model over the constants it becomes clear that the forgetful functor will lift the most types of colimits only for theories without constants, like the theory of affine combinations, for example. Indeed, we show that coproducts and all colimits of diagrams with morphisms reflecting the infinitesimal structure have unique lifts. We conclude the chapter by proving that the categories of infinitesimal models are regular.

The third chapter The third chapter is about applications of the theory of infinitesimal models of algebraic theories to SDG. As a first application we use the smooth Hadamard's lemma to extend the result about Horn formulae-in-context being infinitesimal models of the theory of affine combinations in the syntactic category of $\mathbb{R}$-algebras to the syntactic category of the theory of C^∞-rings, and then further to the opposite category of C^∞-rings. Since smooth manifolds have a fully faithful embedding into this category, we obtain that they are infinitesimal models of the theory of affine combinations and that smooth maps preserve this structure.

The second application concerns well-adapted models. By using the result of Bunge and Dubuc that the ring of line type in every well-adapted model is an archimedean local C^∞-ring, and the explicit presentation of the classifying topos of such rings as the topos of sheaves on the site of finitely presentable C^∞-ring equipped with the Dubuc coverage, we show that every well-adapted model comes with a good supply of infinitesimal models of affine combinations. In particular, every manifold is an infinitesimal model of affine combinations and every smooth map preserves this structure, when embedded into the model.

An important class of geometric spaces in SDG are formal manifolds. We show that a formal manifold becomes an infinitesimal model of the clone of affine combinations by gluing together the respective structures of the infinitesimal models on the charts. In algebraic geometry the counterpart to formal manifolds are schemes. We sketch the proof of how a parallel result can be obtained for schemes.

We conclude this chapter and this thesis by developing some basic infinitesimal geometric algebra in naive SDG. The aim is to give a glance of how infinitesimal geometric algebra can be used to study and develop structures and concepts in SDG. The two kinds of geometric spaces that we introduce are formal 1-manifolds and loci. We show that both types of spaces are infinitesimal models of the theory of affine combinations and that all morphisms become homomorphisms of these models. We then proceed and study the tangent bundle over an infinitesimal model of the theory of affine combinations and develop the linear structure on each tangent space from the infinitesimally affine structure of the space step by step.

Chapter 1

Algebraic Theories

We start by reviewing and comparing three different approaches to formalise the notion of a (one-sorted, finitary) *algebraic theory*. The two approaches relevant to us in this thesis are the syntactic approach using first-order *categorical logic* and *clones*. We also introduce the approach of *Lawvere theories*, since it is closely related to clones.

The main focus will lie on clones, for it is clones that we will use in the next chapter to give the most transparent definition of an infinitesimal model of an algebraic theory. The syntactic approach will be necessary for defining the clones of linear and affine combinations and for the definitions of the various infinitesimalisation of theories in the next chapter. In fact, we shall make substantial use of the categorical logic and syntactic approach introduced here throughout this thesis. We will not make use of Lawvere theories. This is why we only briefly introduce them here in the first chapter, because of their close relationship to clones.

The main purpose of this chapter is to review how the other two approaches are related to clones, and hence why clones may be considered as algebraic theories. Moreover we will introduce the clones of linear and affine combinations, which are crucial for the applications to *Synthetic Differential Geometry* (SDG) in chapter 3. A traditional approach to algebraic theories we will not make any use of in this thesis, and hence have left out completely, is that of finitary monads on Set.

The material covered in this chapter is mostly well-known. The syntactic approach is taken from [Joh02, part D1], Lawvere theories enriched in cartesian closed categories have been studied in [Gra75], and as for clones we use [Gou08, chap. 1], but develop the theory of clones in finite product categories in more depth.

We differ from [Joh02] in that we define cartesian theories purely syntactically. This forces us to adopt a slightly modified version of the sequent calculus and interpretations of formulae-in-context in finite-limit categories. For the sake of being self-contained we give a proof of the soundness theorem for cartesian logic. The proof is, however, merely an adaptation of proofs in [Joh02] and [MR77].

Studying clones in categories with finite products and generalising the well-known equivalence of clones and Lawvere theories to the case of complete and cocomplete cartesian closed categories appears to be new. However, the generalisation is straightforward (but tedious), and might be obvious to experts. The author cannot exclude the possibility that it has been covered in unpublished work or lecture notes already.

The fact that the syntactic category of the theory of (commutative, unital) K-algebras contains the clones of linear and affine combinations seems to be new, and is probably due to the fact that clones in finite-limit categories appear to not have been studied yet.

1.1 Syntactic approach

Algebraic theories are the theories of algebraic structures like groups, rings, modules etc. As such they are theories of equality on terms and traditionally defined syntactically. We shall follow [Joh02, part D1] and define algebraic theories in the framework of *first-order categorical logic*.

At first, it might seem unreasonable to invoke the apparatus of first-order categorical logic for algebraic theories only. However, as we shall make good use of categorical logic in this thesis, we can as well introduce it here. The fragment of first-order logic we will need the most is cartesian logic, which is also the fragment we shall review here. For the additional clauses need for the fragments of coherent and geometric logic, which will use later in chapter 2 and 3, we refer to [Joh02, part D1].

1.1.1 The language of cartesian logic

All of the terminology and definitions are taken from [Joh02, part D1]. We shall, however, adopt a slightly different approach to cartesian theories and cartesian logic from the one developed in [Joh02, part D1].

Definition 1.1.1 (*First-order signature*)**.** A (first-order) **signature** consists of the following data.

- A set of *sorts* Σ-Sort.
- A set of function symbols Σ-Fun, together with a map assigning to each $f \in \Sigma$-Fun its *type*, which consists of a finite non-empty list of sorts. We write

$$f : A_1 \cdots A_n \to B$$

  to indicate that f has type $A_1, \ldots, A_n, B$. (The number n is called the *arity* of f. In the case $n = 0$, f is called a *constant* of sort B.)

- A set Σ-Rel of *relation symbols*, together with a map assigning to each $R \in \Sigma$-Rel its *type*, which consists of a finite list of sorts: we write

$$R \rightarrowtail A_1 \cdots A_n$$

  to indicate that R has type $A_1, \ldots, A_n$. (The number n is called the *arity* of R. In the case $n = 0$, R is called an *atomic proposition*.)

Definition 1.1.2 (Σ-*terms*)**.** Let Σ be a signature, and $X \in [\Sigma\text{-Sort}, \text{Set}]$ a Σ-Sort-set of variables. (Σ-Sort is considered as a discrete category.) The set $T_\Sigma(X)$ of **terms over** Σ is defined recursively by the clauses below. Simultaneously, we define the *sort* of each term and write $t : A$ to denote that t is a term of sort A.

- $x : A$, for $x \in X(A)$;
- $f(t_i, ..., t_n) : B$, if $f : A_1 \cdots A_n \to B$ is a function symbol and $t_1 : A_1, \ldots, t_n : A_n$ are terms.

If f is a constant (that is, a function symbol of arity 0), we commonly write f rather than $f()$ for the term obtained by applying f to the empty string of terms.

If we do not wish to specify the Σ-Sort-set X of variables we will simply speak about (Σ-)terms. In that case we adhere to the tradition of assuming that there is some fixed Σ-Sort-set V so that each $V(A)$ is countably infinite, i.e. there is enough supply of variables for every sort A. We give the definition of *cartesian formulae* [1].

[1] Our notion of 'cartesian formulae' is a pure syntactical one, and hence different from [Joh02, def. D1.3.4], where cartesianess of a formula is defined with respect to a subtheory and invokes the provability of the uniqueness part when existential quantifiers are used.

Definition 1.1.3 (*Cartesian formulae*)**.** Let Σ be a signature. The set F of **cartesian formulae** over Σ is defined recursively by the clauses below, together with, for each formula ϕ, the (finite) set $\mathrm{FV}(\phi)$ of *free variables* of ϕ.

(i) *Relations*: $R(t_1, ..., t_n)$ is in F, if $R \rightarrowtail A_1 \cdots A_n$ is a relation symbol and $t_1 : A, \ldots, t_n : A_n$ are Σ-terms. The free variables of this formula are all the variables occurring in some t_i. (Once again, if R has arity 0 we write simply R rather than $R()$.)

(ii) *Equality*: $s = t$ is in F if s and t are terms of the same sort. $\mathrm{FV}(s = t)$ is the set of variables occurring in s or t (or both).

(iii) *Truth*: $\top$ is in F; $\mathrm{FV}(\top) = \emptyset$.

(iv) *Binary conjunction*: $\phi \wedge \psi$ is in F, if ϕ and ψ are in F. $\mathrm{FV}(\phi \wedge \psi) = \mathrm{FV}(\phi) \cup \mathrm{FV}(\psi)$.

(v) *(Unique) existential quantification* [2]: $(\exists! x)\phi$ is in F, if ϕ is in F and x is a variable. $\mathrm{FV}((\exists! x)\phi) = \mathrm{FV}(\phi) \setminus \{x\}$. Sometimes we may write $(\exists! x : A)\phi$ for $(\exists! x)\phi$ to indicate the sort of the variable x.

Since we restrict ourselves to cartesian logic only in this chapter we shall often omit the adjective 'cartesian' and speak simply of formulae. The following definitions make sense for formulae in any fragment of infinitary first-order logic.

Definition 1.1.4 (*Context*)**.** A **context** is a finite list $\vec{x} = x_1, \ldots, x_n$ of distinct variables. The case $n = 0$ is allowed, being the empty context $[]$. If $\vec{x}$ is a context and y is a variable different from those occuring in $\vec{x}$, then $\vec{x}, y$ will denote the context obtained by appending y to the list $\vec{x}$. Similarly $\vec{x}, \vec{y}$ denotes the result of concatenating contexts $\vec{x}$ and $\vec{y}$ when they are disjoint. The *type* of a context $\vec{x}$ is the string of (not necessarily distinct) sorts of the variables appearing in it. Sometimes we may write the context as $x_1 : A_1, \ldots, x_n : A_n$ to indicate its type.

We say a context $\vec{x}$ is *suitable* for a formula ϕ if all the free variables of ϕ occur in $\vec{x}$. A *formula-in-context* is an expression of the form $\vec{x}.\phi$, where ϕ is a formula and $\vec{x}$ is a suitable context for it. The *canonical context* for ϕ is the context consisting precisely of the distinct free variables of ϕ, listed in the order of their first appearance. Similarly, a *term-in-context* $\vec{x}.t$ is a term t together with a context $\vec{x}$ containing all the variables mentioned in t.

[2] We should warn the reader that this unique existential quantifier is different from the unique existential quantifier in the classical sense. See the discussion in [Cos76].

Remark 1.1.5 (*Substitution*)**.** We shall often make use of the formal operation of substituting terms for variables in a formula-in-context $\vec{x}.\phi$, or a term-in-context $\vec{x}.t$. If $\vec{s} = s_1, \ldots, s_n$ is a list of (not necessarily distinct) terms, of the same length and type as a context $\vec{x}$, then

$$\phi[\vec{s}/\vec{x}] \quad \text{or} \quad t[\vec{s}/\vec{x}]$$

will denote the formula-in-context (well-defined up to α-equivalence, i.e. the renaming of bound variables), or the term-in-context resulting from simultaneously substituting s_i for each free occurrence of x_i in ϕ, respectively, for each occurence in t, for all $1 \leq i \leq n$. In case of ϕ we may have to change the names of bound variables first to avoid capture of variables in $\vec{s}$ by any quantifiers in ϕ. The context after the substitution can be any context that is suitable for each s_i.

Definition 1.1.6 (*Sequent*)**.** A **sequent** over a signature Σ is a formal expression of the form $\phi \vdash_{\vec{x}} \psi$, where ϕ and ψ are formulae over Σ and $\vec{x}$ is a context suitable for both of them.

The intended interpretation of a sequent $\phi \vdash_{\vec{x}} \psi$ is that ψ is a logical consequence of ϕ in the context $\vec{x}$, i.e. that any assignment of individual values to the variables in $\vec{x}$ which makes ϕ true will also make ψ true.

Definition 1.1.7 (*Algebraic theory*)**.** Let Σ be a first-order signature.

(1) A (*presentation of* a) **theory** over Σ is a set $\mathbb{T}$ of sequents over Σ, whose elements are called the (non-logical) *axioms* of $\mathbb{T}$.

(2) Let Σ have one sort, and no relation symbols [3]. (A *presentation of*) an **algebraic theory** is a theory $\mathbb{T}$ over Σ with axioms of the form

$$\top \vdash_{\vec{x}} s = t,$$

where $\vec{x}$ is the canonical context of the (atomic) formula $s = t$.

It is well-known from the example of groups that different axiomatisations of the theory of groups are possible. (See, for example, [Man76, chap. 1.1].) These axiomatisations can differ in signature, but also in the fragment of first-order logic used. A good notion of equivalence of theories is to say that they are equivalent, if the categories of models are equivalent. A stronger notion of equivalence is to require the categories of models to be isomorphic. The advantage of

[3] Like in the definition of cartesian formulae the equality relation is treated separately. We will not list it explicitly as a relation in the signature.

comparing the categories of models is that it can be used to compare different approaches to algebraic theories with each other.

1.1.2 Deduction system

In this section we introduce a deduction system formulated as a sequent calculus. We shall only give the rules of inference for the fragment of cartesian logic only. The full set of rules for infinitary first-order logic (and other fragments) can be found in [Joh02, def. D1.3.1].

The subsequent material is taken from [Joh02, chap. D1.3]. We modify the rules of inference for the existential quantifier to suit its use in cartesian logic. Rules will be written in the form

$$\frac{\Gamma}{\sigma}$$

where Γ is a (possibly empty) list of sequents and σ is a sequent. The intended interpretation of the rule of inference is that if we have established the validity of all the sequents in Γ we may infer the validity of σ. In the particular case when Γ is empty, we shall say that σ is a (logical) *axiom*, and omit the line above it. A *derivation* in the deduction-system will then be a (well-founded) tree with axioms as leaves and the conclusion as its root.

A *proof* or *derivation relative to a theory* $\mathbb{T}$ will be such a tree, except that the leaves are allowed to include (non-logical) axioms of $\mathbb{T}$ as well as the logical axioms, which are the structural rules for manipulating sequents, and the rules for handling the equality predicate. If there exist a derivation of a sequent σ relative to $\mathbb{T}$, we say that ϕ is *provable in* $\mathbb{T}$.

Throughout the subsequent definition, it is assumed that all the sequents which appear are well-formed, i.e. that the contexts which appear in them are suitable for the terms and cartesian formulae on either side.

Definition 1.1.8. The **deduction-system** for cartesian logic contains the following clauses:

(i) (**Structural rules**) The structural rules consist of the *identity axiom*

$$\phi \vdash_{\vec{x}} \phi$$

the *substitution rule*

$$\frac{\phi \vdash_{\vec{x}} \psi}{\phi[\vec{s}/\vec{x}] \vdash_{\vec{y}} \psi[\vec{s}/\vec{x}]}$$

where $\vec{y}$ is any context including all the variables occurring in the string of terms $\vec{s}$, and the *cut rule*

$$\frac{\phi \vdash_{\vec{x}} \psi \qquad \psi \vdash_{\vec{x}} \chi}{\phi \vdash_{\vec{x}} \chi}$$

(ii) (**Equality rules**) The equality rules consist of the axioms

$$\top \vdash_{x} x = x$$

and

$$(\vec{x} = \vec{y}) \wedge \phi \vdash_{\vec{z}} \phi[\vec{y}/\vec{x}]$$

where $\vec{x}$ and $\vec{y}$ are contexts of the same length and type, $(\vec{x} = \vec{y})$ is a shorthand for $((x_1 = y_1) \wedge \cdots \wedge (x_n = y_n))$, and $\vec{z}$ is any context containing $\vec{x}$, $\vec{y}$ and the free variables of ϕ.

(iii) (**Conjuction**) The rules for conjunction are the axioms

$$\phi \vdash_{\vec{x}} \top, \quad \phi \wedge \psi \vdash_{\vec{x}} \phi, \quad \phi \wedge \psi \vdash_{\vec{x}} \psi$$

and the rule

$$\frac{\phi \vdash_{\vec{x}} \psi \qquad \phi \vdash_{\vec{x}} \chi}{\phi \vdash_{\vec{x}} \psi \wedge \chi}$$

(iv) (**Existential quantification**) The rules for existential quantification consist of the rule

$$\frac{\phi \vdash_{\vec{x},y} \psi \qquad \phi \wedge \phi[z/y] \vdash_{\vec{x},y,z} y = z}{(\exists! y)\phi \vdash_{\vec{x}} \psi}$$

and the rules

$$\frac{(\exists! y)\phi \vdash_{\vec{x}} \psi}{\phi \vdash_{\vec{x},y} \psi}, \qquad \frac{(\exists! y)\phi \vdash_{\vec{x}} \psi}{\phi \wedge \phi[z/y] \vdash_{\vec{x},y,z} y = z}$$

(Here our standing hypothesis that all sequents are well-formed includes the information that y is not free in ψ.)

(v) (**Frobenius axiom**)

$$\phi \wedge (\exists! y)\psi \vdash_{\vec{x}} (\exists! y)(\phi \wedge \psi)$$

where y is a variable not in the context $\vec{x}$ (and hence not free in ϕ).

Remark 1.1.9 (*Weakening*)**.** The substitution rule allows the possibility of making the trivial substitution $[\vec{x}/\vec{x}]$. Thus it includes the *weakening rule*, which says that we may derive $\phi \vdash_{\vec{y}} \psi$ from $\phi \vdash_{\vec{x}} \psi$ provided $\vec{y}$ contains all the variables in $\vec{x}$.

Note that we can derive that equality is symmetric and transitive from the rules for equality, the structural rules and the rules for finite conjuction .

1.1.3 Categorical semantics

In the next step we define how to interpret signatures and first-order logic in categories with enough structure. Like in the preceding section we will restrict ourselves to cartesian logic only. The subsequent material is taken from [Joh02, chap. D1.2] (apart from a slight modification of interpretations of cartesian formulae-in-context due to our definition of cartesian theories).

Definition 1.1.10 (Σ-*structures and homomorphisms*)**.** Let Σ be a signature, and C a category with finite products.

(1) A Σ-structure M in C is specified by the following data:

- A function assigning to each sort A in Σ-Sort, an object MA in C. This function is extended to finite strings of sorts by defining $M(A_1, \ldots A_n) = MA_1 \times \ldots \times MA_n$ (and setting $M([])$, where $[]$ denotes the empty string, equal to the terminal object 1 of C).
- A function assigning to each function symbol $f : A_1 \cdots A_n \longrightarrow B$ in Σ-Fun a morphism $Mf : M(A_1, \ldots, A_n) \longrightarrow MB$ in C.
- A function assigning to each relation symbol $R \rightarrowtail A_1 \cdots A_n$ in Σ-Rel a subobject $MR \rightarrowtail M(A_1, \ldots, A_n)$ in C.

(2) The Σ-structures in C are the objects of a category Σ-Str(C) whose morphisms are the Σ-*homomorphisms*. Such a homomorphism $h : M \longrightarrow N$ is specified by a collection of morphisms $h_A : MA \longrightarrow NA$ in C indexed by the sorts of Σ and satisfying the following two conditions:

(i) For each function symbol $f : A_1 \cdots A_n \longrightarrow B$ in Σ-Fun, the diagram

$$\begin{tikzcd}
M(A_1, \ldots, A_n) \arrow[r, "Mf"] \arrow[d, "h_{A_1} \times \ldots \times h_{A_n}"'] & MB \arrow[d, "h_B"] \\
N(A_1, \ldots, A_n) \arrow[r, "Nf"] & NB
\end{tikzcd}$$

commutes.

(ii) For each relation symbol $R \rightarrowtail A_1 \cdots A_n$ in Σ-Rel, there is a commutative diagram in C of the form

$$\begin{tikzcd} MR \arrow[r, tail] \arrow[d] & M(A_1, \ldots, A_n) \arrow[d, "h_{A_1} \times \ldots \times h_{A_n}"] \\ NR \arrow[r, tail] & N(A_1, \ldots, A_n) \end{tikzcd}$$

Identities and composition in Σ-Str(C) are defined componentwise from those in C.

The interpretation of function symbols in a Σ-structure has a natural extension to terms-in-context over Σ.

Definition 1.1.11. Let M be a Σ-structure in a category C with finite products. If $\vec{x}.t$ is a term-in-context over Σ (with $\vec{x} = x_1, \ldots, x_n$, $x_i : A_i$ $(1 \leq i \leq n)$ and $t : B$, say), then a morphism

$$[\![\vec{x}.t]\!] : M(A_1, \ldots, A_n) \to B$$

in C is defined recursively by the following clauses:

- If t is a variable, it is necessarily x_i for some unique $1 \leq i \leq n$, and then $[\![\vec{x}.t]\!] = \mathrm{pr}_i$, the ith product projection.
- If t is $f(t_1, \ldots, t_m)$ (where $t_i : C_i$, say), then $[\![\vec{x}.t]\!]$ is the composite

$$M(A_1, \ldots, A_n) \xrightarrow{([\![\vec{x}.t_1]\!], \ldots, [\![\vec{x}.t_m]\!])} M(C_1, \ldots, C_m) \xrightarrow{Mf} MB$$

  If $m = 0$, i.e., if f is a constant, then the left morphism is to be interpreted as the unique morphism $M(A_1, \ldots, A_n) \to 1$.

For the interpretation of cartesian formulae we require the category C to have finite limits. The intended interpretation of formulae is as subobjects. However, a cartesian formula containing an existential quantifier does not admit an interpretation as a subobject in a finite-limit category, in general. The (sound) use of the existential quantifier is restricted semantically to (provably) unique existence, as indicated by the notation and the respective deduction rule.

There are different ways of dealing with this problem. We define an interpretation for any cartesian formulae, but single out those formulae, which interpretations are *sound*, i.e. do indeed yield monomorphisms.

Definition 1.1.12. Let M be a Σ-structure in a category C with finite limits. A formula-in-context $\vec{x}.\phi$ over Σ (where $\vec{x} = x_1, \ldots, x_n$ and $x_i : A_i$, say) will be interpreted recursively as an (isomorphism class of an) object in $C/M(A_1, \ldots, A_n)$

$$[\![\vec{x}.\phi]\!] \to M(A_1, \ldots, A_n)$$

Simultaneously we define when the interpretation of $\vec{x}.\phi$ is *sound* in M. The recursive clauses are as follows:

(i) If ϕ is $R(t_1, \ldots, t_m)$ where R is a relation symbol (of type $B_1, \ldots, B_m$, say), then $[\![\vec{x}.\phi]\!]$ is the pullback

$$\begin{tikzcd}
{[\![\vec{x}.\phi]\!]} \arrow[r] \arrow[d, tail] & MR \arrow[d, tail] \\
M(A_1, \ldots, A_n) \arrow[r, "{([\![\vec{x}.t_1]\!], \ldots, [\![\vec{x}.t_m]\!])}"] & M(B_1, \ldots, B_m)
\end{tikzcd}$$

and the interpretation is sound. In the case $m = 0$, i.e. when R is an atomic proposition, the bottom morphism is to be interpreted as the unique morphism $M(A_1, \ldots, A_n) \to 1$.

(ii) If ϕ is $s = t$, where s and t are terms of sort B, then $[\![\vec{x}.\phi]\!]$ is the equalizer of

$$\begin{tikzcd}
M(A_1, \ldots, A_n) \arrow[r, shift left, "{[\![\vec{x}.s]\!]}"] \arrow[r, shift right, "{[\![\vec{x}.t]\!]}"'] & M(B)
\end{tikzcd}$$

and is sound in M.

(iii) If ϕ is $\top$, then $[\![\vec{x}.\phi]\!]$ is the top element of $\mathrm{Sub}(M(A_1, \ldots, A_n))$ and the interpretation is sound in M.

(iv) If $[\![\vec{x}.\phi]\!]$ is $\psi \wedge \chi$ then $[\![\vec{x}.\phi]\!]$ is the pullback

$$\begin{tikzcd}
{[\![\vec{x}.\phi]\!]} \arrow[r] \arrow[d] & {[\![\vec{x}.\chi]\!]} \arrow[d] \\
{[\![\vec{x}.\psi]\!]} \arrow[r] & M(A_1, \ldots, A_n)
\end{tikzcd}$$

The interpretation is sound in M, if both $[\![\vec{x}.\psi]\!]$ and $[\![\vec{x}.\chi]\!]$ are sound in M.

(v) If ϕ is $(\exists! y)\psi$ where y is of sort B, then $[\![\vec{x}.\phi]\!]$ is the composite

$$[\![\vec{x}, y.\psi]\!] \longrightarrow M(A_1, \ldots, A_n, B) \xrightarrow{\mathrm{pr}} M(A_1, \ldots, A_n)$$

where pr is the projection on the first n factors. The interpretation is sound in M, if $[\![\vec{x}, y.\psi]\!]$ is sound in M and the above composite is monic.

We say that a Σ-formula-in-context is *sound*, if the formula is sound in M for any Σ-structure M in any category C with finite limits. Every atomic formula (clauses (i) and (ii)) and Horn formula (clauses (i)-(iv)) is sound in any (suitable) context. Only if a cartesian formula has a subformula containing the existential quantifier its interpretation might not be sound.

Definition 1.1.13 (*Models of theories*)**.** Let M be a Σ-structure in a category C.

(1) If $\sigma = (\phi \vdash_{\vec{x}} \psi)$ is a sequent over Σ with $\vec{x}.\phi$ and $\vec{x}.\psi$ having sound interpretations in M, we say σ is **satisfied** in M (and write $M \models \sigma$) if $[\![\vec{x}.\phi]\!] \leq [\![\vec{x}.\psi]\!]$ in $\mathrm{Sub}(M(A_1, \dots, A_n))$.

(2) If $\mathbb{T}$ is a theory over Σ, we say M is a **model** of $\mathbb{T}$ (and write $M \models \mathbb{T}$) if all the axioms of $\mathbb{T}$ are satisfied in M. (In particular, every formula-in-context appearing in $\mathbb{T}$ has a sound interpretation in M.)

(3) We write $\mathbb{T}$-Mod(C) for the full subcategory of Σ-Str(C) whose objects are models of $\mathbb{T}$.

Proposition 1.1.14. *Let C and C' be categories with finite products and $F : C \longrightarrow C'$ a functor that preserves finite limits. F induces a functor $F_* : \mathbb{T}$-Mod$(C) \to \mathbb{T}$-Mod(C').*

Proof. F preserves products and monomorphisms, hence so does the induced functor $F_* : [\Sigma\text{-Sort}, C] \longrightarrow [\Sigma\text{-Sort}, C']$ (by post-composition with F). As such it restricts to a functor Σ-Str$(C) \longrightarrow \Sigma$-Str$(C')$. Since interpretations of formulae-in-context were defined as finite limits, F preserves the interpretations. Furthermore it maps sound interpretations to sound interpretations, and preserves satisfaction, hence restricts to a functor $F_* : \mathbb{T}$-Mod$(C) \longrightarrow \mathbb{T}$-Mod$(C')$ as asserted. □

The category $\mathbb{T}$-Mod(C) is a category over $[\Sigma\text{-Sort}, C]$ with the obvious forgetful functor. The induced functor $F_* : \mathbb{T}$-Mod$(C) \to \mathbb{T}$-Mod(C') commutes with the forgetful functors.

We now turn to the problem of soundness of the deduction system with respect to our definition of the semantics of terms-in-context and formulae-in-context. From now on we shall only work with formulae-in-context that have a sound interpretation. Starting with the structural rules, the only rule that requires attention is the substitution rule. Its soundness is an easy consequence of (2) of the following lemma.

Lemma 1.1.15. (1) (*Substitution in terms*) *Let C be a category with finite products and M a Σ-structure in C. Let $\vec{x}$ be a suitable context for a term $t : C$ (where $x_i : B_i$, say), and let $\vec{s}$ be a list of terms of the same length and type as $\vec{x}$. Further, let $\vec{y}$ be a suitable context for each s_i. Then $[\![\vec{y}.t[\vec{s}/\vec{x}]]\!]$ is the composite*

$$M(A_1, \dots, A_n) \xrightarrow{([\![\vec{y}.s_1]\!], \dots, [\![\vec{y}.s_m]\!])} M(B_1, \dots, B_m) \xrightarrow{[\![\vec{x}.t]\!]} MC$$

(2) (*Substitution in formulae*) *Let C be a category with finite limits and M a Σ-structure in C. Let $\vec{x}.\phi$ be a formula-in-context over Σ with a sound interpretation in M. Let $\vec{s}$ be a list of terms of the same length and type as $\vec{x}$, and let $\vec{y}$ be a context suitable for each s_i. Then $[\![\vec{y}.\phi[\vec{s}/\vec{x}]]\!]$ is sound and can be obtained as a pullback*

$$\begin{tikzcd}
{[\![\vec{y}.\phi[\vec{s}/\vec{x}]]\!]} \arrow[rr] \arrow[d, tail] & & {[\![\vec{x}.\phi]\!]} \arrow[d, tail] \\
M(A_1, \dots, A_n) \arrow[rr, "{([\![\vec{y}.s_1]\!], \dots, [\![\vec{y}.s_m]\!])}"] & & M(B_1, \dots, B_m)
\end{tikzcd}$$

Proof. (1) This is a straightforward induction over the structure of t.

- If t is a variable, it is necessarily of the form $x_i : B_i$, for a unique $1 \leq i \leq n$. In this case $\vec{y}.t[\vec{s}/\vec{x}]$ is $\vec{y}.s_i$. But this is $\mathrm{pr}_i \circ([\![\vec{y}.s_1]\!], \dots, [\![\vec{y}.s_m]\!])$ and $\mathrm{pr}_i = [\![\vec{x}.x_i]\!]$ by definition.
- If t is a constant then the assertion holds true, for precomposing the unique morphism $M(B_1, \dots, B_m) \to 1$ with $([\![\vec{y}.s_1]\!], \dots, [\![\vec{y}.s_m]\!])$ yields the morphism $M(A_1, \dots, A_n) \to 1$.
- If t is $f(t_1, \dots, t_k)$ (with $t_i : C_i$, say) we have that $t[\vec{s}/\vec{x}]$ is $f(t_1[\vec{s}/\vec{x}], \dots, t_k[\vec{s}/\vec{x}])$. By induction hypothesis each $[\![\vec{y}.t_i[\vec{s}/\vec{x}]]\!]$ is the composite

$$M(A_1, \dots, A_n) \xrightarrow{([\![\vec{y}.s_1]\!], \dots, [\![\vec{y}.s_m]\!])} M(B_1, \dots, B_m) \xrightarrow{[\![\vec{x}.t_i]\!]} MC_i$$

Hence $[\![\vec{y}.t[\vec{s}/\vec{x}]]\!]$ is the composite

$$M(A_1, \dots, A_n) \xrightarrow{([\![\vec{y}.s_1]\!], \dots, [\![\vec{y}.s_m]\!])} M(B_1, \dots, B_m) \xrightarrow{([\![\vec{x}.t_1]\!], \dots, [\![\vec{x}.t_k]\!])} M(C_1, \dots, C_k) \xrightarrow{Mf} MC$$

By the definition of the interpretation of $[\![\vec{x}.t]\!]$ this is the composite we wanted to show.

(2) The soundness of $[\![\vec{y}.\phi[\vec{s}/\vec{x}]]\!]$ is a direct consequence of the asserted pullback representation. The pullback diagram in (2) can be shown by induction on the structure of $\vec{x}.\phi$ as well.

- If ϕ is $R(t_1, \dots, t_k)$ (with $t_i : C_i$, say), then $\phi[\vec{s}/\vec{x}]$ is $R(t_1[\vec{s}/\vec{x}], \dots, t_k[\vec{s}/\vec{x}])$. On one hand this has to be interpreted as the pullback

$$\begin{tikzcd}
\llbracket \vec{y}.\phi[\vec{s}/\vec{x}] \rrbracket \arrow[r] \arrow[d, tail] & MR \arrow[d, tail] \\
M(A_1, \dots, A_n) \arrow[r, "(\llbracket \vec{y}.t_1[\vec{s}/\vec{x}] \rrbracket, \dots, \llbracket \vec{y}.t_k[\vec{s}/\vec{x}] \rrbracket)"] & M(C_1, \dots, C_k)
\end{tikzcd}$$

On the other hand we can apply (1) to the bottom morphism and obtain $\llbracket \vec{y}.\phi[\vec{s}/\vec{x}] \rrbracket$ by pasting together the two pullbacks

$$\begin{tikzcd}
\llbracket \vec{y}.\phi[\vec{s}/\vec{x}] \rrbracket \arrow[r] \arrow[d, tail] & \llbracket \vec{x}.\phi \rrbracket \arrow[r] \arrow[d, tail] & MR \arrow[d, tail] \\
M(A_1, \dots, A_n) \arrow[r, "(\llbracket \vec{y}.s_1 \rrbracket, \dots, \llbracket \vec{y}.s_m \rrbracket)"] & M(B_1, \dots, B_m) \arrow[r, "(\llbracket \vec{x}.t_1 \rrbracket, \dots, \llbracket \vec{x}.t_k \rrbracket)"] & M(C_1, \dots, C_k)
\end{tikzcd}$$

The left diagram is the desired pullback diagram. The argument still applies if R is an atomic proposition.

- If ϕ is the formula $(t_1 = t_2)$ then $\phi[\vec{s}/\vec{x}]$ is $t_1[\vec{s}/\vec{x}] = t_2[\vec{s}/\vec{x}]$. We apply (1) to the terms and form the pullback X

$$\begin{tikzcd}
X \arrow[r] \arrow[d, tail] & \llbracket \vec{x}.\phi \rrbracket \arrow[d, tail] \\
M(A_1, \dots, A_n) \arrow[r, "(\llbracket \vec{y}.s_1 \rrbracket, \dots, \llbracket \vec{y}.s_m \rrbracket)"] \arrow[dr, shift left, "\llbracket \vec{y}.t_1[\vec{s}/\vec{x}] \rrbracket"] \arrow[dr, shift right, "\llbracket \vec{y}.t_2[\vec{s}/\vec{x}] \rrbracket"'] & M(B_1, \dots, B_m) \arrow[d, shift right, "\llbracket \vec{x}.t_2 \rrbracket"'] \arrow[d, shift left, "\llbracket \vec{x}.t_1 \rrbracket"] \\
& M(C)
\end{tikzcd}$$

Any morphism that equalises $\llbracket \vec{y}.t_1[\vec{s}/\vec{x}] \rrbracket$ and $\llbracket \vec{y}.t_2[\vec{s}/\vec{x}] \rrbracket$ equalises $\llbracket \vec{x}.t_1 \rrbracket$ and $\llbracket \vec{x}.t_2 \rrbracket$, hence factors uniquely through $\llbracket \vec{x}.\phi \rrbracket$. But then it also factors uniquely through X, for it is a pullback. This shows X the equaliser of $\llbracket \vec{y}.t_1[\vec{s}/\vec{x}] \rrbracket$ and $\llbracket \vec{y}.t_2[\vec{s}/\vec{x}] \rrbracket$, and hence that $X \cong \llbracket \vec{y}.\phi[\vec{s}/\vec{x}] \rrbracket$.

- If ϕ is the formula $\psi \wedge \chi$ then $\phi[\vec{s}/\vec{x}]$ is $\psi[\vec{s}/\vec{x}] \wedge \chi[\vec{s}/\vec{x}]$. Applying the interpretation of a conjuction to $\vec{y}.\phi[\vec{s}/\vec{x}]$ and the induction hypothesis to ψ and χ yields the diagram

$$\begin{tikzcd}
& {[\![\vec{y}.\phi[\vec{s}/\vec{x}]]\!]} \arrow[rr, dashed] \arrow[dl, tail] \arrow[dd, tail] \arrow[dddl, tail, dashed] & & {[\![\vec{x}.\phi]\!]} \arrow[dl, tail] \arrow[dd, tail] \arrow[dddl, tail, dashed] \\
{[\![\vec{y}.\psi[\vec{s}/\vec{x}]]\!]} \arrow[rr] \arrow[dd, tail] & & {[\![\vec{x}.\psi]\!]} \arrow[dd, tail] & \\
& {[\![\vec{y}.\chi[\vec{s}/\vec{x}]]\!]} \arrow[rr] \arrow[dl, tail] & & {[\![\vec{x}.\chi]\!]} \arrow[dl, tail] \\
M(A_1, \dots, A_n) \arrow[rr, dashed, "{([\![\vec{y}.s_1]\!],\dots,[\![\vec{y}.s_m]\!])}"] & & M(B_1, \dots, B_m) &
\end{tikzcd}$$

where the top dashed arrow is due to $[\![\vec{x}.\phi]\!]$ being a pullback. We want to show the square formed by the dashed arrows a pullback. The front and bottom squares are pullbacks by induction hypothesis. The left square is a pullback by construction. A simple diagram chase then shows that the dashed square is a pullback.

- If ϕ is the formula $(\exists! z)\psi$ then $\phi[\vec{s}/\vec{x}]$ is $(\exists! z)\psi[\vec{s}/\vec{x}]$. Assuming that the pullback representation holds for $\vec{x}, z.\psi$ (with the $m+1$-term being the variable z), we obtain the diagram

$$\begin{tikzcd}
{[\![\vec{y}.\phi[\vec{s}, z/\vec{x}, z]]\!]} \arrow[r] \arrow[d, tail] & {[\![\vec{x}, z.\phi]\!]} \arrow[d, tail] \\
M(A_1, \dots, A_n, B) \arrow[r, "{([\![\vec{y}.s_1]\!],\dots,[\![\vec{y}.s_m]\!],[\![\vec{y}.z]\!])}"] \arrow[d, "\mathrm{pr}"] & M(B_1, \dots, B_m, B) \arrow[d, "\mathrm{pr}"] \\
M(A_1, \dots, A_n) \arrow[r, "{([\![\vec{y}.s_1]\!],\dots,[\![\vec{y}.s_m]\!])}"] & M(B_1, \dots, B_m)
\end{tikzcd}$$

By induction hypothesis the upper square is a pullback. Since $[\![\vec{y}.z]\!]$ is the projection onto the factor MB the lower square is a pullback. This shows the outer square a pullback as asserted.

$\square$

As for the soundness of the equality rules, we only need to pay attention to the second rule. Let $A_1, \dots, A_n$ be the type of the contexts $\vec{x}$ and $\vec{y}$. To simplify notation we will write A for $M(A_1, \dots, A_n)$ and write the context in the rule as the concatenation $\vec{y}, \vec{x}, \vec{z}$. (Here the context $\vec{z}$ contains in particular all the free variables in ϕ that are not in $\vec{x}$.) We will write B for

$M(B_1, \ldots, B_m)$, where $B_1, \ldots, B_m$ is the type of $\vec{z}$. By the substitution lemma 1.1.15 and the interpretations of intersections and equations we obtain two pullback diagrams

$$\begin{tikzcd}
[\![\vec{y}, \vec{x}, \vec{z}.\phi[\vec{y}/\vec{x}]]\!] \arrow[r] \arrow[d, tail] & [\![\vec{y}, \vec{x}, \vec{z}.\phi]\!] \arrow[d, tail] & [\![\vec{y}, \vec{x}, \vec{z}.(\vec{x} = \vec{y}) \wedge \phi]\!] \arrow[l, tail] \arrow[d, tail] \\
A \times A \times B \arrow[r, "(\mathrm{pr}_2, \mathrm{pr}_2, \mathrm{pr}_3)"] & A \times A \times B & A \times B \cong [\![\vec{y}, \vec{x}, \vec{z}.(\vec{x} = \vec{y})]\!] \arrow[l, tail, "\Delta_A \times 1_B"']
\end{tikzcd}$$

where pr_i denotes the projection on the ith factor and Δ_A the diagonal map. Now $(\mathrm{pr}_2, \mathrm{pr}_2, \mathrm{pr}_3) \circ (\Delta_A \times 1_B) = (\Delta_A \times 1_B)$, hence $[\![\vec{y}, \vec{x}, \vec{z}.(\vec{x} = \vec{y}) \wedge \phi]\!]$ factors through $[\![\vec{y}, \vec{x}, \vec{z}.\phi[\vec{y}/\vec{x}]]\!]$ since the left square is a pullback.

The soundness of the conjuction rules is a straightforward consequence of the interpretation of conjuctions as pullbacks and the interpretation of $\vec{x}.\top$ as the maximal element in the respective subobject in the meet-semilattice of subobjects.

We show the soundness of the rules of the existential quantifier. By the substitution lemma 1.1.15, if ψ does not depend on y, then the subsequent diagram is a pullback

$$\begin{tikzcd}
[\![\vec{x}, y.\psi]\!] \arrow[r] \arrow[d, tail] & [\![\vec{x}.\psi]\!] \arrow[d, tail] \\
M(A_1, \ldots, A_n, B) \arrow[r, "\mathrm{pr}"] & M(A_1, \ldots, A_n)
\end{tikzcd}$$

It follows easily that $[\![\vec{x}, y.\phi]\!] \leq [\![\vec{x}, y.\psi]\!]$ implies $[\![\vec{x}.(\exists! y)\phi]\!] \leq [\![\vec{x}.\psi]\!]$ (and vice versa), if the interpretation of the existential quantifier is sound. We show that this is the case for the first rule.

Lemma 1.1.16. *Let M be Σ-structure in a finite-limit category C. If $[\![\vec{x}, y.\phi]\!]$ is sound in M, and if*

$$M \models (\phi \wedge \phi[z/y] \vdash_{\vec{x},y,z} (y = z))$$

then $[\![\vec{x}.(\exists! y)\phi]\!]$ is sound.

Proof. (Cf. [MR77, lem. 2.4.3].) We simplify the notation and write A for $M(A_1, \ldots, A_n)$, B for $M(B)$ and

$$(a, b) : R \rightarrowtail A \times B$$

for $[\![\vec{x}, y.\phi]\!]$. With this notation

- $[\![\vec{x}.(\exists! y)\phi]\!]$ becomes $R \xrightarrow{a} A$,

- $[\![\vec{x}, y, z.\phi]\!]$ becomes $R \times B \xrightarrow{(a,b)\times 1_B} A \times B \times B$,
- $[\![\vec{x}, y, z.\phi[z/y]]\!]$ becomes $R \times B \xrightarrow{(a\,\mathrm{pr}_1, \mathrm{pr}_2, b\,\mathrm{pr}_1)} A \times B \times B$, and
- $[\![\vec{x}, y, z.(y = z)]\!]$ becomes $A \times B \xrightarrow{1_A \times \Delta_B} A \times B \times B$.

We have to show a a monomorphism. For any two morphisms $f, g : Z \to R$ such that $af = ag$, the morphisms $(f, bg), (g, bf) : Z \to R \times B$ make the following square commutative

$$\begin{tikzcd}
Z \arrow[rr, "{(f,bg)}"] \arrow[dr, dashed] \arrow[dd, "{(g,bf)}"'] & & R \times B \arrow[dd, tail, "{(a,b)\times 1_B}"] \\
& A \times B \arrow[dr, tail, "1_A \times \Delta_B"] & \\
R \times B \arrow[rr, tail, "{(a\,\mathrm{pr}_1,\mathrm{pr}_2,b\,\mathrm{pr}_1)}"] & & A \times B \times B
\end{tikzcd}$$

and hence factor uniquely through the pullback of $(a\,\mathrm{pr}_1, \mathrm{pr}_2, b\,\mathrm{pr}_1)$ along $(a, b) \times \mathrm{pr}_2$. But this pullback factors through $1_A \times \Delta_B$ (by assumption), hence so does Z, as indicated by the dashed arrow. The morphisms bf and bg must be equal. Since a and b are jointly monic, we obtain $f = g$. This shows a a monomorphism as asserted.

$\square$

It remains to show the soundness of the third rule. Using the notation from the previous proof, for a monomorphism a in the pullback square

$$\begin{tikzcd}
Z \arrow[rr, tail, "{(v_1,v_2)}"] \arrow[dr, dashed] \arrow[dd, tail, "{(u_1,u_2)}"'] & & R \times B \arrow[dd, tail, "{(a,b)\times 1_B}"] \\
& A \times B \arrow[dr, tail, "1_A \times \Delta_B"] & \\
R \times B \arrow[rr, tail, "{(a\,\mathrm{pr}_1,\mathrm{pr}_2,b\,\mathrm{pr}_1)}"] & & A \times B \times B
\end{tikzcd}$$

we find that $u_1 = v_1$ and $u_2 = v_2 = bu_1$. Hence both composites factor through $1_A \times \Delta_B$ as indicated. This establishes the soundness of the third rule.

The soundness of the Frobenius axiom is a consequence of the following pullback diagram

$$\begin{tikzcd}
[\![\vec{x}, y.\phi \wedge \psi]\!] \arrow[r, tail] \arrow[d, tail] & [\![\vec{x}, y.\psi]\!] \arrow[d, tail] \\
[\![\vec{x}, y.\phi]\!] \arrow[r, tail] \arrow[d] & M(A_1, \ldots, A_n, B) \arrow[d, "\mathrm{pr}"] \\
[\![\vec{x}.\phi]\!] \arrow[r, tail] & M(A_1, \ldots, A_n)
\end{tikzcd}$$

The bottom square is a pullback (by weakening), the top square by definition. This shows the outer square a pullback. As subobjects of $M(A_1, \ldots, A_n)$ the outer pullback square can be interpreted as either $[\![\vec{x}.\phi \wedge (\exists! y)\psi]\!]$ or $[\![\vec{x}.(\exists! y)(\phi \wedge \psi)]\!]$, hence $[\![\vec{x}.\phi \wedge (\exists! y)\psi]\!] \cong [\![\vec{x}.(\exists! y)(\phi \wedge \psi)]\!]$, which shows the soundness of the Frobenius axiom (and its converse). We have shown the following theorem.

Theorem 1.1.17 (Soundness of cartesian logic)**.** *Let $\mathbb{T}$ be a cartesian theory over a signature Σ, C a category with finite limits, and M a $\mathbb{T}$-model in C. If σ is a* (*cartesian*) *sequent that is provable in $\mathbb{T}$, then $M \models \sigma$.*

Since we know that atomic and Horn formulae-in-context are sound the soundness theorem gives us the means to define a sound use of the existential quantifier relative to a theory $\mathbb{T}$ in purely logical terms: a formula-in-context $\vec{x}.(\exists! y)\phi$ is **sound relative to** $\mathbb{T}$, if $\vec{x}, y.\phi$ is either a Horn formula, or is sound relative to $\mathbb{T}$, and

$$\phi \wedge \phi[z/y] \vdash_{\vec{x},y,z} y = z$$

is provable in $\mathbb{T}$. This is essentially the definition of a cartesian formula-in-context relative to $\mathbb{T}$ given in [Joh02, def. D1.3.4(a)].

Remark 1.1.18 (*Models of algebraic theories in finite-product categories*)**.** The disadvantage of our approach is that we have only defined models of algebraic theories in finite-limit categories, whereas the minimal structure of a category required to be able to define models of an algebraic theory is that of finite products. Such a definition of a model/algebra can be obtained as follows.

For an algebraic theory $\mathbb{T}$ the signature Σ contains one sort and function symbols only. We have defined Σ-structures and the interpretation of terms-in-context in categories with finite products. The axioms of $\mathbb{T}$ do, strictly speaking, involve equalisers. However, the axioms actually state that certain terms are equal (in the canonical context of the equation), which can be interpreted as the corresponding morphisms being equal. Exactly as in the finite-limit case a product-preserving functor $F : C \to C'$ has a lift along the forgetful functors to $F_* : \mathbb{T}\text{-Mod}(C) \to \mathbb{T}\text{-Mod}(C')$ for an algebraic theory $\mathbb{T}$.

The following operations, which follow from the structural and equality rules in a finite-limit category, are still sound in finite-product categories: We can substitute terms-in-context in a term-in-context for variables and weaken contexts of terms. Substitution of terms in an equation that is satisfied yields again an equation that is satisfied. Applying an n-ary function symbol f to n equations $s_i = t_i$ that are satisfied yields another equation $f(t_1, \ldots, t_n) = f(s_1, \ldots, s_n)$ that is satisfied.

Finite-product categories only matter in this chapter for the comparison of the syntactic approach with that of clones. In later chapters we will always work in finite-limit categories when using categorical logic.

1.1.4 Finitely generated free $\mathbb{T}$−algebras

We review the syntactically constructed (universal) finitely generated models of an algebraic theory in Set. They are important for the comparison of the syntactic approach with that of clones in Set.

Proposition 1.1.19 (Term algebras)**.** *Let Σ be a signature with one sort (A, say) and function symbols only. Let X be a set. The set of terms $T_\Sigma(X)$ carries a natural Σ-structure that maps A to $T_\Sigma(X)$ and each $f \in \Sigma\text{-Fun}$ (of arity n, say) to the map*

$$T_\Sigma(X)^n \longrightarrow T_\Sigma(X), \qquad (t_1, \ldots, t_n) \mapsto f(t_1, \ldots, t_n)$$

The Σ-structure $T_\Sigma(X)$ (together with the inclusion of variables $\iota : X \to T_\Sigma(X)$ is the free Σ-structure over the set X:

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "f"'] & UT_\Sigma(X) \arrow[d, "Uh"] \\ & UM(A) \end{tikzcd} \qquad \begin{tikzcd} T_\Sigma(X) \arrow[d, dashed, "\exists! h"] \\ M(A) \end{tikzcd}$$

where $U : \Sigma\text{-Str(Set)} \to \text{Set}$ is the forgetful functor.

Proof. The homomorphism h is defined recursively on the structure of terms in the obvious way. See, for example, [AR94, prop. 3.2] or [Man76, 1.20]. □

Lemma 1.1.20. *Let $\mathbb{T}$ be an algebraic theory over a signature Σ and $V = \{x_1, x_2, \dots\}$ the fixed countably infinite set of variables. There is a smallest equivalence relation E on $T_\Sigma(V)$ generated by $\mathbb{T}$, such that E is a Σ-substructure of $T_\Sigma(V) \times T_\Sigma(V)$ and E is closed under substitution. More explicitly, E is the smallest equivalence relation on $T_\Sigma(V)$ satisfying*

(1) *For any axiom $\top \vdash_{\vec{x}} s = t$ of $\mathbb{T}$, (s, t) is in E.*

(2) *For any (s, t) in E, for any suitable context $\vec{x}$ for the formula $s = t$ and any list of terms $\vec{s}$ with the same length as $\vec{x}$, $(s[\vec{s}/\vec{x}], t[\vec{s}/\vec{x}])$ is in E.*

(3) *For any n-ary function symbol f and $(s_i, t_i) \in E$, $1 \leq i \leq n$, $(f(t_1, \dots, t_n), f(s_1, \dots, s_n))$ is in E.*

Proof. The total equivalence relation $T_\Sigma(V) \times T_\Sigma(V)$ has these properties, and the properties are stable under intersections. E is the intersection of all equivalence relations on $T_\Sigma(V)$ that satisfy (1)-(3). (See also [Bor94a, lem. 3.2.5].) □

For each $n \in \mathbb{N}$ we define $T_\Sigma(n)$ as the Σ-substructure $T_\Sigma(x_1, \dots, x_n) \rightarrowtail T_\Sigma(V)$ of Σ-terms in the context $x_1, \dots, x_n$. In particular, $T_\Sigma(0)$ is the set of the Σ-terms build from constants and function symbols only. Let E_n be the restriction of E to $T_\Sigma(n)$. Define $F_\mathbb{T}(n)$ as the quotient $T_\Sigma(n)/E_n$.

Proposition 1.1.21 (Finitely generated free $\mathbb{T}$-algebras)**.** *For each $n \in \mathbb{N}$ there is a natural Σ-structure on $F_\mathbb{T}(n)$ making it the free $\mathbb{T}$-model over the set $\{x_1, \dots, x_n\}$, respectively over $\emptyset$ for $n = 0$.*

Proof. See [Bor94a, lem. 3.26.6 and 3.2.8]. Despite the case $n = 0$ not being considered explicitly in [Bor94a] the given proofs extend naturally to this case as well. □

1.2 Enriched Lawvere Theories

Lawvere theories constitute a categorical approach to algebraic theories. They have been introduced by Lawvere in [Law63]. An algebraic theory is taken to be a category C with finite products and objects the finite powers of one distinguished object, A say. The morphisms $A^n \to A$ are the n-ary operations of the theory. The category of C-models in a finite-product category D is the full subcategory of the functor category $[C, D]$ of finite-product preserving functors.

For our purposes we will need the notion of Lawvere theories enriched in a complete and cocomplete cartesian closed category with finite colimits. Lawvere theories enriched in certain symmetric monoidal closed categories have been studied by Borceux and Day in [BD80]. Their work generalises the work of Gray in [Gra75] on Lawvere theories enriched in a complete and cocomplete cartesian closed category. Since [Gra75] appears to be not accessible anymore, we use the definition given in [BD80], apart from choosing a different category for arities, which corresponds to the one used for (ordinary) Lawvere theories. A more recent and more general account of enriched Lawvere theories is given in [LW16].

Let $\mathcal{V}$ be a cartesian closed category, which is complete and cocomplete. For the basic notions of enriched categories we refer to [Kel82] and [Bor94a]. We only spell out the definition of finite (conical) $\mathcal{V}$-products in a $\mathcal{V}$-category A, since it is central to the notion of $\mathcal{V}$-enriched Lawvere theories. In the language of enriched categories, a finite *(conical) $\mathcal{V}$-product* in a $\mathcal{V}$-category A is the $\mathcal{V}$-limit of a $\mathcal{V}$-functor $F : n \longrightarrow A$ weighted by the $\mathcal{V}$-functor $\Delta_1 : A \longrightarrow \mathcal{V}$, which maps each object of A to the terminal object 1 in $\mathcal{V}$. The category n is the discrete category with n-objects considered as the $\mathcal{V}$-category with $n(j, j) = 1$ and $n(j, k) = 0$ for the objects j, k in n. (0 denotes the initial object in $\mathcal{V}$.)

The $\mathcal{V}$-limit of F weighted by Δ_1 is saying that there is an object p in A and a $\mathcal{V}$-natural isomorphism in $a \in \operatorname{ob} A$

$$\mathcal{V}\text{-Nat}(\Delta_1, A(a, F(-))) \cong A(a, p)$$

By [Bor94a, lem. 6.3.3] there is a bijective correspondence between morphisms $v \longrightarrow A(a, p)$ and $\mathcal{V}$-natural transformations $\Delta_1 \longrightarrow [v, A(a, F(-))]$ for each $v \in \operatorname{ob}\mathcal{V}$ and $a \in \operatorname{ob} A$. But since Δ_1 is constant 1, and since the $\mathcal{V}$-functor $F : n \longrightarrow A$ is equivalent to giving a family of n objects $a_i = F(i)$ in A, such $\mathcal{V}$-natural transformations are families of n morphisms $f_i : v \longrightarrow A(a, a_i)$; hence $A(a, p) \cong \prod_{i=1}^{n} A(a, a_i)$, $\mathcal{V}$-natural in a, as to be expected.

In particular, if we substitute p for a, then we get a morphism

$$1 \xrightarrow{1_p} A(p, p) \longrightarrow \prod_{i=1}^{n} A(p, a_i)$$

which corresponds to n morphisms $\pi_i^n : 1 \longrightarrow A(p, a_i)$. We call these morphisms projections. They are indeed the product projections that make p an (ordinary) product in the underlying category A_0. Moreover, the underlying functor $F_0 : A_0 \longrightarrow A'_0$ of a $\mathcal{V}$-functor $F : A \longrightarrow A'$ that preserves finite $\mathcal{V}$-products, is finite-product preserving. We summarise these observations in a lemma.

Lemma 1.2.1. *Let A be a $\mathcal{V}$-category. An object p is a $\mathcal{V}$-product of $a_i \in \mathrm{ob}\, A$, $1 \leq i \leq n$, if and only if there is a $\mathcal{V}$-natural isomorphism*

$$A(-, p) \cong \prod_{i=1}^{n} A(-, a_i)$$

A $\mathcal{V}$-functor $F : A \to B$ preserves $\mathcal{V}$-products if and only if for each $\mathcal{V}$ product $p = a_1 \times \ldots \times a_n$ there is a $\mathcal{V}$-natural isomorphism $B(-, F(p)) \cong \prod_{i=1}^{n} B(-, F(a_i))$ and the subsequent square commutes for all $a \in \mathrm{ob}\, A$

$$\begin{tikzcd}
A(a, p) \arrow[r, "\cong"] \arrow[d, "F_{a,p}"] & \prod_{i=1}^{n} A(a, a_i) \arrow[d, "\prod_{i=1}^{n} F_{a,a_i}"] \\
B(F(a), F(p)) \arrow[r, "\cong"] & \prod_{i=1}^{n} B(F(a), F(a_i))
\end{tikzcd}$$

Let Fin be a skeleton of the category of finite sets. We can make it into a $\mathcal{V}$-category by defining the hom-objects $\mathrm{Fin}(n, m)$ by the copowers $\mathrm{hom}_{\mathrm{Fin}}(n, m)1$. Since the underlying category Fin has finite coproducts, the $\mathcal{V}$-category Fin^{op} has finite $\mathcal{V}$-products. Indeed, products distribute over coproducts in the cartesian closed category $\mathcal{V}$, and thus $(X \times Y)1 \cong (X1) \times (Y1)$. This yields

$$\mathrm{Fin}^{op}(m, n) = \mathrm{hom}_{\mathrm{Fin}}(n, m)1 \cong (\mathrm{hom}_{\mathrm{Fin}}(1, m)^n)1 \cong (\mathrm{hom}_{\mathrm{Fin}}(1, m)1)^n = \mathrm{Fin}^{op}(m, 1)^n$$

which is $\mathcal{V}$-natural in m. By lemma 1.2.1 this shows each object n an n-fold $\mathcal{V}$-product of 1 in Fin^{op}. The $\mathcal{V}$-category Fin^{op} will serve as the category of arities.

Definition 1.2.2 (*Enriched Lawvere theory*)**.** Let $\mathcal{V}$ be a cartesian closed category, which is complete and cocomplete.

(1) A **$\mathcal{V}$-Lawvere theory** (L, α) is $\mathcal{V}$-category with finite $\mathcal{V}$-products together with a $\mathcal{V}$-functor $\alpha : \mathrm{Fin}^{op} \to L$ that preserves $\mathcal{V}$-products and is a bijection on objects.

(2) A **morphism of $\mathcal{V}$-Lawvere theories** $F : (L, \alpha) \to (L', \alpha')$ is a $\mathcal{V}$-product preserving $\mathcal{V}$-functor $F : L \to L'$ such that $F\alpha = \alpha'$. The category of $\mathcal{V}$-Lawvere theories and morphisms of such will be denoted by $\mathrm{Law}(\mathcal{V})$.

(3) A **model** M of a $\mathcal{V}$-Lawvere theory L is a $\mathcal{V}$-product preserving $\mathcal{V}$-functor $M : L \to \mathcal{V}$. *Homomorphisms* of L-models $h : M \to M'$ are $\mathcal{V}$-natural transformations. The $\mathcal{V}$-category of models L-Mod is the full $\mathcal{V}$-subcategory of the $\mathcal{V}$-category $[L, \mathcal{V}]$ of $\mathcal{V}$-product preserving functors.

When there is no danger of confusion, then we will denote the objects in Fin^{op} and L by $n \in \mathbb{N}$, which stands for $\alpha(n)$. Since a morphism of $\mathcal{V}$-Lawvere theories $F : (L, \alpha) \longrightarrow (L', \alpha')$ satisfies $F\alpha = \alpha'$ we write this as if it were the identity on objects, i.e. $F(n) = n$.

Remark 1.2.3. Let (L, α) be a $\mathcal{V}$-Lawvere theory. The pair of underlying functors (L_0, α_0) is a Set-Lawvere theory, which is a Lawvere theory as defined in [Law63].

1.3 Clones and their algebras

The syntactic approach to universal algebra via signatures and equational presentations has the disadvantage that the notion of algebraic theory is not sufficiently well-represented as a mathematical object itself. This makes it difficult and cumbersome to study and compare algebraic theories per se. Lawvere theories resolve this problem, if one adopts an approach using category theory. Universal algebra has found another solution to this problem in the notion of a *clone*.

Our interest in clones is for an entirely different reason. We want to consider clones as structures representing algebraic theories in categories with finite products, so that we can define infinitesimal models of algebraic theories in general, and infinitesimally affine spaces, groups and vector spaces, in particular. The aim of this section is thus to define clones and their algebras as structures in finite-product categories and to give two important examples of clones: the clone of an algebraic theory, which we will need for exhibiting the equivalence of the syntactic approach and that of clones, and the clone of affine combinations over a (commutative) ring-object in a finite-product category, which we will need for the definition of infinitesimally affine spaces in SDG.

1.3.1 Clones in a category with finite products

In universal algebra a *clone* is a subset of all n-ary operations $f : A^n \longrightarrow A$ over a fixed set A containing all the projections and being closed under (multi-)composition ([Coh81, chap. III.3], [Beh14, def. 2.1]). By abstracting from the particular set A to the algebraic structure of the operations themselves we arrive at the notion of an *abstract clone*, which appears to have been introduced by Philip Hall [Coh81, exercise III.3.3]. Our definition is a generalisation of [Gou08, def. 1.2.1] to categories with finite products [4].

[4] The normalisation axiom is not explicitly stated in [Gou08, def. 1.2.1], but it seems to have been assumed implicitly. For example, normalisation is used in [Gou08, lem. 1.5.1], for the clones $K_{\Phi,E}$ satisfy normalisation and normalisation is stable under isomorphisms of clones.

Definition 1.3.1 (*Clone*)**.** Let C be a category with finite products. A **clone** O over C has the following data:

- For every $n \in \mathbb{N}$ an object $O(n)$.
- For every $(n, k) \in \mathbb{N}^2$ a morphism $*_{nk} : O(n) \times O(k)^n \to O(k)$.
- For every $n \geq 1$ and $1 \leq j \leq n$ global elements $\pi_j^n : 1 \to O(n)$.

The subsequent diagrams are rendered commutative:

(1) (**Associativity**) For every triple $(n, m, k) \in \mathbb{N}^3$

$$\begin{tikzcd}
& O(n) \times O(m)^n \times (O(k)^m)^n \arrow[dr, "\cong"] & \\
O(n) \times O(m)^n \times O(k)^m \arrow[ur, "1_{O(n)} \times 1_{O(m)^n} \times \Delta^m_{O(k)}"] \arrow[d, "*_{nm} \times 1_{O(k)^m}"'] & & O(n) \times (O(m) \times O(k)^m)^n \arrow[d, "1_{O(n)} \times (*_{mk})^n"] \\
O(m) \times O(k)^m \arrow[dr, "*_{mk}"'] & & O(n) \times O(k)^n \arrow[dl, "*_{nk}"] \\
& O(k) &
\end{tikzcd}$$

where $\Delta^m_{O(k)} : O(k)^m \to (O(k)^m)^n$ is the diagonal map.

(2) (**Projection**) For every $n \geq 1$, $m \in \mathbb{N}$ and $1 \leq j \leq n$

$$\begin{tikzcd}
1 \times O(m)^n \arrow[r, "\pi_j^n \times 1_{O(m)^n}"] \arrow[d, "\cong"'] & O(n) \times O(m)^n \arrow[d, "*_{nm}"] \\
O(m)^n \arrow[r, "\mathrm{pr}_j"] & O(m)
\end{tikzcd}$$

where $\mathrm{pr}_j : O(m)^n \to O(m)$ denotes the projection onto the jth factor.

(3) (**Unit**) For each $n \geq 1$

$$\begin{tikzcd}
O(n) \times O(n)^n \arrow[r, "*_{nn}"] & O(n) \\
O(n) \times 1 \arrow[u, "1_{O(n)} \times (\pi_1^n, \ldots, \pi_n^n)"] \arrow[r, "\cong"] & O(n) \arrow[u, "1_{O(n)}"']
\end{tikzcd}$$

(4) (**Normalisation**)

$$\begin{tikzcd} O(0) \times 1 \arrow[r, "*_{00}"] \arrow[d, "\cong"'] & O(0) \\ O(0) \arrow[ur, "1_{O(n)}"'] & \end{tikzcd}$$

The isomorphisms appearing in the diagrams are the natural coherence isomorphisms of the symmetric monoidal category $(C, \times, 1)$. Another way of constructing them is to apply the universal property of products to the respective equivalent construction of (iterated) products.

From the viewpoint of universal algebra the morphisms $*_{n0} : O(n) \times O(0)^n \longrightarrow O(0)$ evaluate n-ary operations on constants resulting in another constant. The morphisms [5] $*_{0k} : O(0) \longrightarrow O(k)$ name the k-ary operations that are constant. Normalisation $*_{00} = 1_{O(0)}$ makes sure that each constant names itself. For the cases $n = 0$ and/or $m = 0$ the associativity axiom ensures that naming and evaluation on constants are compatible.

Definition 1.3.2 (*Clone homomorphism*)**.** Let O and O' be clones in C. A **clone homomorphism** is a family of C-morphisms $f_n : O(n) \longrightarrow O'(n)$, $n \in \mathbb{N}$, rendering the following diagrams commutative:

(1) For every $(n, k) \in \mathbb{N}^2$

$$\begin{tikzcd} O(n) \times O(k)^n \arrow[r, "f_n \times (f_k)^n"] \arrow[d, "*_{nk}"] & O'(n) \times O'(k)^n \arrow[d, "*'_{nk}"] \\ O(k) \arrow[r, "f_k"] & O'(k) \end{tikzcd}$$

(2) For every $n \geq 1$ and $1 \leq j \leq n$

$$\begin{tikzcd} & 1 \arrow[dl, "\pi_j^n"'] \arrow[dr, "\pi_j'^n"] & \\ O(n) \arrow[rr, "f_n"] & & O'(n) \end{tikzcd}$$

Clones together with clone homomorphisms form a category $\mathrm{clone}(C) \hookrightarrow [\mathbb{N}, C]$, where $\mathbb{N}$ is considered as a discrete category.

[5] We shall use the same name for morphisms $f : X \times 1 \longrightarrow Y$ and $g : X \longrightarrow Y$, if $f = g\rho_X$ for the natural coherence isomorphism $\rho_X : X \times 1 \cong X$. The same shall apply to morphisms $f : 1 \times X \longrightarrow Y$ and $g : X \longrightarrow Y$, if $f = g\lambda_X$ for the coherence isomorphism $\lambda_X : 1 \times X \cong X$.

Proposition 1.3.3. *Let C and C' be categories with finite products, and $F : C \to C'$ a finite-product preserving functor. F induces a functor $F_* : \mathrm{clone}(C) \to \mathrm{clone}(C')$ with $F_*(O)(n) = F(O(n))$.*

Proof. Every functor $F : C \to C'$ induces a functor $F_* : [\mathbb{N}, C] \to [\mathbb{N}, C']$ by post-composition. As a finite-product preserving functor F preserves the commutative diagrams in definition 1.3.1, commutes with products, projections, the terminal object, and the coherence isomorphisms. Hence F_* restricts to a functor $F_* : \mathrm{clone}(C) \to \mathrm{clone}(C')$. □

Remark 1.3.4 (*Clones and operads*)**.** We have introduced clones O as functors $\mathbb{N} \to C$ where $\mathbb{N}$ is taken to be a discrete category. Using the projections we can lift O to a functor from the category Fin (a skeleton of the category of finite sets) to C by assigning to a map of finite sets $f : n \to m$ the morphism $O(f)$ defined as the composite

$$O(n) \xrightarrow{\rho^{-1}_{O(n)}} O(n) \times 1 \xrightarrow{1_{O(n)} \times (\pi^m_{f(1)}, \dots, \pi^m_{f(n)})} O(n) \times O(m)^n \xrightarrow{*_{nm}} O(m)$$

where $\rho_{O(n)} : O(n) \times 1 \cong O(n)$ denotes the natural coherence isomorphism. In the case $n = 0$ the morphism $O(f)$ is just $*_{0m} \circ \rho^{-1}_{O(n)}$. The functoriality of O is a consequence of the associativity and unit axiom. Furthermore, with this definition the morphisms

$$*_{nm} : O(n) \times O(m)^n \to O(m)$$

become natural in m and dinatural in n. In the case $C = \mathrm{Set}$ the unit axiom gives us a further natural transformation $\mathrm{Fin}(1, -) \to O$. Together with the associativity, projection and unit axioms such functors are called *cartesian operads* in [Hyl16] and [Tri13]. The precise relationship of clones in Set and operads is studied, for example, in [Tro02], who takes a more functorial approach to operads, and in [Gou08] who adopts the multicategory point of view.

1.3.2 O-Algebras

Clones have been introduced as 'abstract clones' encoding the structure of syntactically defined operations of finite arity. The notion of *algebra of a clone* introduces the corresponding notion of a model/representation of the syntactically defined operations.

Definition 1.3.5 (*Algebra of a clone*)**.** Let O be a clone in C. An O**-algebra** (A, α) is a C-object A together with morphisms

$$\alpha_n : O(n) \times A^n \to A$$

for each $n \in \mathbb{N}$ rendering the subsequent diagrams commutative:

(1) (**Associativity**) For every pair $(n, m) \in \mathbb{N}^2$

$$\begin{tikzcd}
& O(n) \times O(m)^n \times (A^m)^n \arrow[dr, "\cong"] & \\
O(n) \times O(m)^n \times A^m \arrow[ur, "1_{O(n)} \times 1_{O(m)^n} \times \Delta_{A^m}"] \arrow[d, "*_{nm} \times 1_{A^m}"'] & & O(n) \times (O(m) \times A^m)^n \arrow[d, "1_{O(n)} \times (\alpha_m)^n"] \\
O(m) \times A^m \arrow[dr, "\alpha_m"'] & & O(n) \times A^n \arrow[dl, "\alpha_n"] \\
& A &
\end{tikzcd}$$

where $\Delta_{A^m} : A^m \to (A^m)^n$ is the diagonal map.

(2) (**Projection**) For every $n \geq 1$, $1 \leq j \leq n$

$$\begin{tikzcd}
1 \times A^n \arrow[r, "\pi_j^n \times 1_{A^n}"] \arrow[d, "\cong"'] & O(n) \times A^n \arrow[d, "\alpha_n"] \\
A^n \arrow[r, "\mathrm{pr}_j"] & A
\end{tikzcd}$$

where $\mathrm{pr}_j : A^n \to A$ denotes the projection onto the jth factor.

Definition 1.3.6 (*O-algebra homomorphism*)**.** Let O be a clone in C and (A, α), (A', α') two O-algebras. An O**-algebra homomorphism** is a C-morphism $h : A \to A'$ rendering the following diagram commutative for every $n \in \mathbb{N}$

$$\begin{tikzcd}
O(n) \times A^n \arrow[r, "1_{O(n)} \times h^n"] \arrow[d, "\alpha_n"] & O(n) \times (A')^n \arrow[d, "\alpha'_n"] \\
A \arrow[r, "h"] & A'
\end{tikzcd}$$

O-algebras and O-algebra homomorphisms form a category $O\text{-}\mathrm{Alg}(C)$ over C.

Proposition 1.3.7. *Let C be a category with finite products. A clone homomorphism $f : O \to O'$ induces a functor $f^* : O'\text{-Alg}(C) \to O\text{-Alg}(C)$ of categories over C*

$$\begin{tikzcd} O'\text{-Alg}(C) \arrow[rr, "f^*"] \arrow[dr, "U"'] & & O\text{-Alg}(C) \arrow[dl, "U"] \\ & C & \end{tikzcd}$$

The map $f \mapsto f^*$ is functorial in $\mathrm{clone}(C)^{op}$, i.e. $(fg)^* = g^* \circ f^*$ and $1_O^* = I_{O\text{-Alg}(C)}$. In particular, clone isomorphisms induce isomorphisms of the corresponding categories of algebras.

Proof. Let (A, α) be an O'-algebra. Define $f^*\alpha$ to be the family of C-morphisms $\alpha_n \circ (f_n \times 1_{A^n})$, and define $f^*(A, \alpha)$ as $(A, f^*\alpha)$. $(A, f^*\alpha)$ is an O-algebra. Associativity and projection hold, because they do for α and because f renders commutative the diagrams in definition 1.3.2. In light of the definition of $f^*\alpha$ a C-morphism $h : A \to A'$ that is an O'-algebra homomorphism $h : (A, \alpha) \to (A', \alpha')$ is easily seen an O-algebra homomorphism $(A, f^*\alpha) \to (A', f^*\alpha')$. The construction is clearly functorial as asserted. □

Proposition 1.3.8. *Let C and C' be categories with finite products, O a clone in C, and $F : C \to C'$ a finite-product preserving functor. F induces a functor $F_* : O\text{-Alg}(C) \to F_*(O)\text{-Alg}(C')$ with $F_*(A, \alpha) = (F(A), F(\alpha))$.*

Proof. By proposition 1.3.3 F induces a functor $F_* : \mathrm{clone}(C) \to \mathrm{clone}(C')$. In particular, the componentwise defined $F_*(O)$ is a clone in C'. As a finite-product preserving functor F preserves the commutative diagrams in definition 1.3.5, preserves products, projections and the terminal object. For any O-algebra (A, α), $(F(A), F(\alpha))$ is thus readily seen an $F_*(O)$-algebra, and for every morphism f of O-algebras $F(f)$ is easily seen a homomorphism of the respective $F_*(O)$-algebras. □

1.3.3 Free O-algebras

Let O be a clone in a category C with finite products. We fix $k \in \mathbb{N}$. The associativity axiom for k and the projection axiom (for $m = k$) in definition 1.3.1 show that the morphisms

$$*_{nk} : O(n) \times O(k)^n \to O(k)$$

make $O(k)$ into an O-algebra. Intuitively, $O(k)$ should be the 'free O-algebra on k generators'. For arbitrary categories C with finite products we can only make sense of this for the case $k = 0$.

Proposition 1.3.9. *Let O be clone in C. The O-algebra $(O(0), *_{(-)0})$ is an initial object in the category O-$\mathrm{Alg}(C)$.*

Proof. Let (A, α) be an O-algebra. We consider the associativity diagram for the O-algebra A in definition 1.3.5 for $m = 0$

$$\begin{tikzcd}
 & O(n) \times O(0)^n \times (A^0)^n \arrow[dr, "\cong"] & \\
O(n) \times O(0)^n \times A^0 \arrow[ur, "1_{O(n)} \times 1_{O(0)^n} \times \Delta_{A^0}"] \arrow[d, "*_{n0} \times 1_{A^0}"'] & & O(n) \times (O(0) \times A^0)^n \arrow[d, "1_{O(n)} \times (\alpha_0)^n"] \\
O(0) \times A^0 \arrow[dr, "\alpha_0"] & & O(n) \times A^n \arrow[dl, "\alpha_n"'] \\
 & A. &
\end{tikzcd}$$

Since $A^0 = 1$ and $\Delta_{A^0} = 1_1$ after applying the natural coherence isomorphisms $X \times 1 \cong X$ at the four appropriate vertices we obtain by the universal property of products (or MacLane's coherence theorem [Mac63][thm. 4.2])

$$\begin{tikzcd}
O(n) \times O(0)^n \arrow[rr, "1_{O(n)} \times 1_{O(0)^n}"] \arrow[d, "*_{n0}"'] & & O(n) \times O(0)^n \arrow[d, "1_{O(n)} \times (\alpha_0)^n"] \\
O(0) \arrow[dr, "\alpha_0"] & & O(n) \times A^n \arrow[dl, "\alpha_n"'] \\
 & A. &
\end{tikzcd}$$

Hence $\alpha_0 : O(0) \to A$ is an O-algebra homomorphism. Any O-algebra homomorphism $f : O(0) \to A$ renders commutative the diagram

$$\begin{tikzcd}
O(0) \arrow[r, "1_{O(0)}"] \arrow[d, "*_{00}"'] & O(0) \arrow[d, "\alpha_0"] \\
O(0) \arrow[r, "f"] & A
\end{tikzcd}$$

The normalisation condition in definition 1.3.1 yields $f = \alpha_0$. □

For C = Set the O-algebras $(O(k), *_{(-)k})$ are indeed free on k generators. We rephrase the definition of a clone and its algebras in terms of elements before we give a proof.

Remark 1.3.10. In the case of C = Set the commutativity of the diagrams in definition 1.3.1 is equivalent with the satisfaction of the following sets of equations:

- (**Associativity**) For every $\sigma \in O(n)$, $t_1, \ldots, t_n \in O(m)$, $s_1, \ldots s_m \in O(k)$

  $$\sigma *_{nk} (t_1 *_{mk} (s_1, \ldots, s_m), \ldots, t_n *_{mk} (s_1, \ldots, s_m)) = (\sigma *_{nm} (t_1, \ldots, t_n)) *_{mk} (s_1, \ldots, s_m).$$

  In particular, naming and evaluation on constants are compatible:

  - In the case $m = 0$ the t_i are constants and the associativity states

    $$\sigma *_{nk} (t_1, \ldots, t_n) = *_{0k}(\sigma *_{n0} (t_1, \ldots, t_n))$$

  - In the case $n = 0$ the operation σ is a constant and the associativity states

    $$*_{0k}(\sigma) = (*_{0m}(\sigma)) *_{mk} (s_1, \ldots, s_m)$$

- (**Projection**) For every $n \geq 1$, $1 \leq j \leq n$, $t_1, \ldots, t_n \in O(m)$

  $$\pi^n_j *_{nm} (t_1, \ldots, t_n) = t_j$$

- (**Unit**) For each $\sigma \in O(n)$, $n \geq 1$

  $$\sigma *_{nn} (\pi^n_1, \ldots, \pi^n_n) = \sigma$$

- (**Normalisation**) $*_{00}(c) = c$.

Similar sets of equations hold for the associativity and projection axioms of O-algebras. An $\mathbb{N}$-indexed family of maps $f_n : O(n) \to O'(n)$ constitutes a *clone homomorphism* $f : O \to O'$, if $f_n(\pi^n_j) = \pi'^n_j$ for all $n \geq 1$ and $1 \leq j \leq n$, and for all $n, m \in \mathbb{N}$, $\sigma \in O(n)$ and $t_1, \ldots, t_n \in O(m)$

$$f_m(\sigma *_{nm} (t_1, \ldots, t_n)) = f_n(\sigma) *'_{nm} (f_m(t_1), \ldots, f_m(t_n))$$

The case $n = 0$ is to be interpreted as above. A map $h : A \longrightarrow A'$ is an *O-algebra homomorphism* $h : (A, \alpha) \longrightarrow (A', \alpha')$, if for $n \geq 1$ and $\sigma \in O(n)$

$$h(\alpha_n(\sigma, (a_1, \ldots, a_n))) = \alpha'_n(\sigma, (h(a_1), \ldots, h(a_n))),$$

and $h(\alpha_0(c)) = \alpha'_0(c)$ for $c \in O(0)$.

Remark 1.3.11 (*The cartesian theory of clones*)**.** The presentation of Set-clones in terms of elements in the previous remark can be used to define a cartesian theory $\mathbb{T}$ of clones as a *many-sorted algebraic theory*. The signature Σ has a sort $O(n)$ for each $n \in \mathbb{N}$, a function symbol $*_{nm} : O(n)O(m) \cdots O(m) \longrightarrow O(m)$ of arity $n + 1$ for every pair $(n, m) \in \mathbb{N}^2$ with $n \geq 1$, a unary function symbol $*_{0m} : O(0) \longrightarrow O(m)$ for each $m \in \mathbb{N}$ and constants $\pi^n_j : \longrightarrow O(n)$ for each $n \in \mathbb{N}$ and $1 \leq j \leq n$. There are no relation symbols and the axioms of $\mathbb{T}$ are the equations given in the previous remark in their respective canonical contexts. Simply by comparing the definitions it is clear that clones in a finite product category C are exactly the models of $\mathbb{T}$, and that clone homomorphisms are exactly the Σ-homomorphisms. In other words, we have $\text{clone}(C) = \mathbb{T}\text{-Mod}(C)$ up to renaming. (As for one-sorted algebraic theories the axioms can be interpreted as equality of morphisms, hence many-sorted algebraic theories can be interpreted in finite-product categories as well.)

Proposition 1.3.12 (*Finitely generated free O-algebras*)**.** *Let O be a clone in* Set*. The O-algebra $(O(k), *_{(-)k})$ has the universal property*

$$\begin{tikzcd} \{1, \ldots, k\} \arrow[r, "\{\pi^k_1, \ldots, \pi^k_k\}"] \arrow[dr, "h"'] & O(k) \arrow[d, "f"] & & (O(k), *_{(-)k}) \arrow[d, dashed, "\exists! f"] \\ & A & & (A, \alpha) \end{tikzcd}$$

Proof. With remark 1.3.10 we can see easily the map

$$f : O(k) \longrightarrow A, \quad \sigma \mapsto \alpha_k(\sigma, (h(1), \ldots, h(k)))$$

to be the unique O-algebra map making the diagram commute. The uniqueness is a consequence of the unit axiom for $O(k)$ and f being an O-algebra homomorphism. As for the existence the associativity of α shows that the f defined above is indeed an O-algebra homomorphism. □

Remark 1.3.13. If we rephrase the proof of the preceding proposition in terms of diagrams, then we can apply it to the case of a finite-product category C. In that case we obtain a bijection of hom-sets $\mathcal{O}\text{-Alg}(\mathcal{O}(k), A) \cong \mathcal{C}(1, A^k)$, natural in A. The correspondence is given by

$$f \mapsto (f \circ \pi_1^k, \ldots, f \circ \pi_k^k) \quad \text{with inverse} \quad (a_1, \ldots, a_n) \mapsto \alpha_k \circ (1_{\mathcal{O}(k)} \times (a_1, \ldots, a_n)) \circ \rho^{-1},$$

where $\rho : \mathcal{O}(k) \times 1 \cong \mathcal{O}(k)$ is the natural coherence isomorphism. If $\mathcal{C}$ has finite coproducts then replacing the set $\{1, \ldots, k\}$ with the copower $k \cdot 1$ in the above diagram will give the corresponding universal property for $\mathcal{O}(k)$ in $\mathcal{C}$.

1.3.4 Examples

Examples of clones studied in this section are the endomorphism clones, clones of algebraic theories and the initial clones. We shall also prove that the category of algebras for a clone of an algebraic theory is isomorphic to the category of models for that theory. The clones of linear and affine combinations will be studied in the next section.

Examples 1.3.14 (*Endomorphism clones*)**.**

(a) For any object A in $\mathcal{C}$ we obtain an **endomorphism clone** $\mathrm{End}(A)$ in Set as follows [Gou08, example 1.1.2]

- $\mathrm{End}(A)(n) = \mathcal{C}(A^n, A)$,
- for $n \geq 1$ the maps $*_{nk} : \mathcal{C}(A^n, A) \times \mathcal{C}(A^k, A)^n \to \mathcal{C}(A^k, A)$ are given by (multi-)composition

$$(g, (f_1, \ldots, f_n)) \mapsto g \circ (f_1, \ldots, f_n),$$

  and the maps $*_{0k} : \mathcal{C}(1, A) \to \mathcal{C}(A^k, A)$ map constants $1 \to A$ to the composites $A^k \to 1 \to A$,
- π_j^n names the projection $\mathrm{pr}_j : A^n \to A$ of A^n onto its jth factor.

The axioms of a clone are a consequence of the associativity of the composition in $\mathcal{C}$ and the universal property of products. They can be readily verified using remark 1.3.10.

(b) Let $\mathcal{C}$ be a cartesian closed category and A and object in $\mathcal{C}$. Generalising the previous example we obtain an endomorphism clone $\underline{\mathrm{End}}(A)$ in $\mathcal{C}$

- $\underline{\mathrm{End}}(A)(n) = [A^n, A]$,
- the maps $*_{nk} : [A^n, A] \times [A^k, A]^n \to [A^k, A]$ are given by (multi-)composition

$$[A^n, A] \times [A^k, A]^n \xrightarrow{1_{[A^n,A]} \times \iota} [A^n, A] \times [A^k, A^n] \xrightarrow{\circ} [A^k, A]$$

where $\iota : [A^k, A]^n \cong [A^k, A^n]$ is a natural isomorphism and '$\circ$' denotes the internal composition.

- $\pi^n_j : 1 \to [A^n, A]$ is the exponential transpose of the projection $\mathrm{pr}_j : A^n \to A$.

The associativity axiom is a consequence of the associativity of the internal composition, the projection and unit axiom follow from the π^n_j being the exponential transposes of projections, and the internal composition being defined via the exponential transpose of successive evaluation morphisms. The normalisation axiom is a direct consequence of the definition of $*_{00}$. In this case $\iota : 1 \cong [1, 1]$ is the exponential transpose of the identity map 1_1. Another way of seeing $\underline{\mathrm{End}}(A)$ a clone is to use the technique of generalised elements and combine it with remark 1.3.10. This amounts to showing that $\hom(X, \underline{\mathrm{End}}(A))$ is a clone in Set natural in $X \in \mathrm{ob}\, C$. With the Yoneda lemma we can then conclude that it is a clone in C. To see $\hom(X, \underline{\mathrm{End}}(A))$ a clone requires us to rewrite the $*_{nk}$ in terms of evaluation morphisms.

Remark 1.3.15. Let C be a cartesian closed category, O a clone and $A \in \mathrm{ob}\, C$. The product-hom adjunction yields bijections

$$\frac{\alpha_n : O(n) \times A^n \longrightarrow A}{\hat{\alpha}_n : O(n) \longrightarrow [A^n, A]}$$

for each $n \in \mathbb{N}$. These bijections together induce a bijection between clone morphisms $\hat{\alpha} : O \to \underline{\mathrm{End}}(A)$ and O-algebra structures (A, α) on A. The associativity diagrams of an O-algebra (A, α) correspond thereby to the diagrams (1) in definition 1.3.2 for $\hat{\alpha}$, and the projection diagrams to the diagrams (2) in definition 1.3.2 (and vice versa).

Remark 1.3.16. The correspondence in remark 1.3.15 suggests a way of defining O-algebras for a Set-clone O in a finite-product category C. Let A be an object in C. An O-algebra structure on A is a clone homomorphism $\alpha : O \to \mathrm{End}(A)$ [Gou08, def. 1.2.4]. An O-algebra homomorphism $h : (A, \alpha) \to (A', \alpha')$ in C is a C-morphism $h : A \to A'$ rendering commutative the diagrams

$$\begin{tikzcd} A^n \arrow[r, "h^n"] \arrow[d, "\alpha_n(\sigma)"'] & (A')^n \arrow[d, "\alpha'_n(\sigma)"] \\ A \arrow[r, "h"] & A' \end{tikzcd}$$

for all $n \in \mathbb{N}$ and $\sigma \in O(n)$ [Gou08, def. 1.2.5]. In the case of $C = \mathrm{Set}$ this definition is clearly equivalent to definition 1.3.2. We shall denote the category of O-algebras in C by $O\text{-Alg}(C)$ again. It will be clear from the context whether O is a C-clone, or a Set-clone, and hence which definition we mean.

Examples 1.3.17 (*Clones of algebraic theories*)**.**

(a) Let $\mathbb{T}$ be an algebraic theory over the signature Σ, and M a $\mathbb{T}$-model in a category C with finite products. The intersection of (small) families of subclones of $\mathrm{End}(MA)$ in $[\mathbb{N}, \mathrm{Set}]$ is again a subclone. We can thus define the subclone of $\mathrm{End}(MA)$ generated by the operations $Mf : MA^n \to MA$, $f \in \Sigma$-Fun. In the case of $C = \mathrm{Set}$ (and apart from the constants) this is a *clone of operations* as it is usually defined in universal algebra. (See e.g. [Coh81, chap. III.3], [KPS14, def. 2.1], [Beh14, def. 2.1].)

(b) (**Clone of** $\mathbb{T}$) We can also assign a clone $O_{\mathbb{T}}$ to the algebraic theory $\mathbb{T}$ directly. Define $O_{\mathbb{T}}(n)$ as the finitely generated free $\mathbb{T}$-algebra $F_{\mathbb{T}}(n) = T_\Sigma(n)/E_n$ (see lemma 1.1.20). Each t in the term-algebra $T_\Sigma(n)$ can be considered a term-in-context $\vec{x}.t$ with $\vec{x} = x_1, \ldots, x_n$. The operation of substitution induces maps $*_{nm} : T_\Sigma(n) \times T_\Sigma(m)^n \to T_\Sigma(m)$. Due to property (2) in lemma 1.1.20 any $(s,t) \in E_n$ will be mapped to E_m by $*_{nm}$, i.e.

$$(s *_{nm} (s_1, \ldots, s_n), t *_{nm} (s_1, \ldots, s_n)) \in E_m$$

for any $\vec{s} \in T_\Sigma(m)^n$. Because of property (3) in lemma 1.1.20 this remains true if we replace $\vec{s}$ by any $\vec{t}$ such that (s_i, t_i) is in E_m for each $1 \leq i \leq n$. The map $*_{nm}$ thus descends to a map

$$*_{nm} : F_{\mathbb{T}}(n) \times F_{\mathbb{T}}(m)^n \to F_{\mathbb{T}}(m), \qquad ([t], ([s_1], \ldots, [s_n])) \mapsto [t[s_1/x_1, \ldots, s_n/x_n]]$$

The π^n_j name the (equivalence classes of) variables $[x_j] \in F_{\mathbb{T}}(n)$ for $n \geq 1$. The axioms of a clone follow from the corresponding properties of substitution of terms. Regarding normalisation in particular, since $F_{\mathbb{T}}(0)$ is the set of (equivalence classes of) terms with no free variables, substitution becomes the identity map.

For the empty theory $\emptyset$ over Σ the $\emptyset$-algebra $F_\emptyset(n)$ is $T_\Sigma(n)$. For any theory $\mathbb{T}$ over Σ the family of quotient maps $q_n : T_\Sigma(n) \to F_{\mathbb{T}}(n)$ yields a clone homomorphism $q : O_\emptyset \to O_{\mathbb{T}}$.

Proposition 1.3.18. *Let $\mathbb{T}$ be an algebraic theory over Σ and $O_{\mathbb{T}}$ its clone in* Set. *For any category C with finite products the categories* $\mathbb{T}$-Mod(C) *and* $O_{\mathbb{T}}$-Alg(C) *are isomorphic as categories over C.*

Proof. It is easy to see that any clone homomorphism $\alpha : O_\emptyset \to \mathrm{End}(A)$ is equivalent to a Σ-structure M on A. $M_\alpha(f)$ is defined by $\alpha_n(f(x_1, \ldots, x_n))$ for an n-ary function symbol $f \in \Sigma$-Fun. Conversely, given a Σ-structure M definition 1.1.11 and the substitution lemma 1.1.15(1) together show that mapping a term $t \in T_\Sigma(n)$ to its interpretation $[\![x_1, \ldots, x_n.t]\!]$ yields a clone homomorphism $\alpha_M : O_\emptyset \to \mathrm{End}(A)$. The two construction are mutually inverse.

In light of these two constructions an $O_\emptyset$-algebra homomorphism is the same as a Σ-homomorphism. Rephrased in categorical terms we obtain an isomorphism of $O_\emptyset$ -Alg(C) and Σ-Str(C) that commutes with the respective forgetful functors to C:

$$\begin{tikzcd} O_\emptyset\text{-Alg}(C) \arrow[rr, "\cong"] \arrow[dr, "U"'] & & \Sigma\text{-Str}(C) \arrow[dl, "U"] \\ & C & \end{tikzcd}$$

By remark 1.1.18 a Σ-structure M in C is a $\mathbb{T}$-model, if for every axiom of $\mathbb{T}$ the interpretations of the respective terms-in-context in M are equal. This means that the clone homomorphism α_M factors through $q : O_\emptyset \to O_\mathbb{T}$. Conversely, by precomposing a clone homomorphism $\alpha : O_\mathbb{T} \to \mathrm{End}(A)$ with q the Σ-structure $M_{\alpha\circ q}$ is a $\mathbb{T}$-model. We see that the above isomorphism of categories restricts to an isomorphism of the full subcategories $O_\mathbb{T}$ -Alg(C) and $\mathbb{T}$ -Mod(C). □

Example 1.3.19 (*Initial clones*)**.** We consider the algebraic theory $\mathbb{T}$ of equality, i.e. the empty theory over the signature Σ with only one sort and no function symbols. The term algebra $T_\Sigma(n)$ is the finite set $\{x_1, \dots, x_n\}$. The induced clone $O_\mathbb{T}$ can be described as follows: it assigns a finite set n to each natural number n and the maps $*_{nm} : n \times m^n \to m$ are the unique maps $\{\mathrm{pr}_1, \dots, \mathrm{pr}_n\}$ induced by the universal property of the coproduct

$$\begin{tikzcd} m^n + \ldots + m^n \arrow[r, "\exists!\{\mathrm{pr}_1,\ldots,\mathrm{pr}_n\}"] & m \\ m^n \arrow[u, "i_j"] \arrow[ur, "\mathrm{pr}_j"'] & \end{tikzcd}$$

Here we have used that $n \times m^n$ is the nth copower of m^n in Set. The i_j denotes the coproduct injection into the jth component and pr_j the projection onto the jth factor.

For any other clone O in Set the universal property of the coproduct yields for each n and the n-element set of the π^n_j a unique map

$$\{\pi^n_1, \dots, \pi^n_n\} : n \to O(n)$$

It is easy to check that this is a clone homomorphism. Moreover, (2) in definition 1.3.2 implies that this clone homomorphism is unique. This exhibits $O_\mathbb{T}$ as the initial object in the category clone(Set).

Let C be a *distributive category*, i.e. C has finite products and coproducts, and the coproducts distribute over the products in a canonical way (the canonical morphism from coproducts of products to the respective products of coproducts is an isomorphism). Using the copowers in C we can repeat the construction above in C and define a clone O_I. To see that O_I is indeed a clone in C we can use proposition 1.3.3. Indeed, the clone $O_{\mathbb{T}}$ for the theory of equality is a clone in the category of finite sets. Mapping a finite set X to its copower $X1$ in C induces a functor from the category of finite sets to C. Since C is distributive this functor preserves finite products. Using the description above it is clear that $O_{\mathbb{T}}$ is mapped to O_I by this functor. Like $O_{\mathbb{T}}$ in Set the clone O_I is an initial object in the category clone(C). The unique morphism from O_I to any C-clone O is constructed like in Set.

Remark 1.3.20. Let C be an *infinitary distributive* category, i.e. C is a distributive category with small coproducts and the small coproducts distribute over finite products as well. In this case taking copowers of the terminal object induces a functor $F : \mathrm{Set} \to C$ that preserves finite products. By proposition 1.3.3 we obtain a functor $F_* : \mathrm{clone}(\mathrm{Set}) \to \mathrm{clone}(C)$. So every Set-clone is represented by a clone in C.

Let (A, α) be an $F_*(O)$-algebra. Since coproducts distribute over products, $F_*(O)(n) \times A^n$ is isomorphic to the copower $O(n)A^n$. Every α_n hence corresponds to an $O(n)$-indexed family of morphisms $A^n \to A$; in other words, α_n corresponds to a map $f_n : O(n) \to C(A^n, A)$. The family of the maps f_n constitutes a clone homomorphism $O \to \mathrm{End}(A)$ in Set. Conversely, by the universal property of copowers a clone homomorphism $f : O \to \mathrm{End}(A)$ yields an $F_*(O)$-algebra structure α_f on A. This bijection between $F_*(O)$-algebra structures on A and clone homomorphisms $O \to \mathrm{End}(A)$ can be seen in the same way as the correspondence in remark 1.3.15: the associativity diagrams for α correspond to the diagrams (1) in definition 1.3.2 for f, and the projection diagrams to the diagrams (2) in definition 1.3.2 (and vice versa).

In remark 1.3.16 we have defined O-algebras in C of a Set-clone O as the clone homomorphisms $f : O \to \mathrm{End}(A)$. For infinitary distributive categories (like Grothendieck toposes, for example) we see that (A, f) is an O algebra if and only if (A, α_f) is an $F_*(O)$-algebra. Furthermore, it is not difficult to see that this correspondence extends to the homomorphisms as well. The categories $F_*(O)\text{-Alg}(C)$ and $O\text{-Alg}(C)$ are thus isomorphic as categories over C.

1.3.5 The clones of linear and affine combinations

The last two examples we shall discuss in this chapter are *clones of linear and affine combinations* in a finite-limit category C with a K-algebra object. They will be important for the applications in SDG later on; in particular, for the definition of an *infinitesimally affine space*.

Let $\mathbb{T}$ be the (presentation of a one-sorted) *algebraic theory of (commutative, unital) K-algebras* for a fixed (commutative, unital) ring K: the signature Σ has one sort A, two binary function symbols '$+$' and '$\cdot$', a unary function symbol for each $k \in K$, a unary function symbol '$-$', and the constants 0 and 1. The axioms are the usual axioms for a commutative ring together with axioms for each $k \in K$ stating that k is an endomorphism of the underlying abelian group, $\top \vdash_x kx = (k1) \cdot x$, 0_K is the constant 0-map and 1_K is the identity map.

Recall that the **syntactic category** $C_{\mathbb{T}}$ of $\mathbb{T}$ [Joh02, chap. D1.4] is the category with objects the α-equivalence classes of (cartesian) formulae-in-context $\{\vec{x}.\phi\}$ (equivalent up to the renaming of all variables, not just the bound ones) and morphisms $\{\vec{x}.\phi\} \to \{\vec{y}.\psi\}$ the $\mathbb{T}$*-provable equivalence classes* of formulae-in-context $[\vec{x}, \vec{y}.\theta]$ that are $\mathbb{T}$*-provably functional*. $\mathbb{T}$-provable equivalence means that $[\theta] = [\theta']$, if both sequents $\theta \vdash_{\vec{x},\vec{y}} \theta'$ and $\theta' \vdash_{\vec{x},\vec{y}} \theta$ are provable in $\mathbb{T}$. For θ to be $\mathbb{T}$-provably functional the following two sequents must be provable in $\mathbb{T}$:

$$\theta \vdash_{\vec{x},\vec{y}} \phi \wedge \psi,$$
$$\phi \vdash_{\vec{x}} (\exists! \vec{y})\theta$$

(All formulae-in-context are assumed to be sound relative to $\mathbb{T}$. $\exists!\vec{y}$ stands for $(\exists! y_1) \cdots (\exists! y_n)$.) Let A be the (unique) sort in Σ. The intuition is that $\{\vec{x}.\phi\}$ stands for the set of elements of A^n that satisfy ϕ, where n is the length of the context. Morphisms are maps between such sets, which are described by a (definable) functional relation. We will write A^n for $\{\vec{x}.\top\}$, where the context $\vec{x}$ has length n (with $A^0 = 1$ for the empty context[6]).

The category $C_{\mathbb{T}}$ has finite limits [Joh02, lem. D1.4.2]. We define the **clone $\mathcal{L}$ of linear combinations** in $C_{\mathbb{T}}$ as follows:

- $\mathcal{L}(n) = A^n$
- $*_{nm} : A^n \times (A^m)^n \to A^m$ is the $\mathbb{T}$-provable equivalence class of the formula-in-context

$$\vec{\lambda}, \vec{x}^1, \ldots, \vec{x}^n, \vec{y}.(\vec{y} = \lambda_1 \vec{x}^1 + \ldots + \lambda_n \vec{x}^n)$$

  (Where $\vec{y} = \lambda_1 \vec{x}^1 + \ldots + \lambda_n \vec{x}^n$ stands for the conjunction $\bigwedge_{1 \le j \le m} (y_j = \lambda_1 x_j^1 + \ldots + \lambda_n x_j^n)$. Note that the k in x_j^k stands for an index and not a power k.)

[6]Whether we mean 1 as the terminal object, or as the constant of the unit in Σ will be clear from the context.

- $\pi_j^n : 1 \longrightarrow A^n$ is the $\mathbb{T}$-provable equivalence class of the formula-in-context

$$\vec{y}.(\vec{y} = (0, \ldots, 0, \underset{j}{1}, 0, \ldots, 0))$$

  meaning the 1 is at the jth position. ($\vec{y} = (0, \ldots, 0, 1, 0, \ldots, 0)$ is again a shorthand for $(y_1 = 0) \wedge \ldots \wedge (y_{j-1} = 0) \wedge (y_j = 1) \wedge (y_{j+1} = 0) \wedge \ldots \wedge (y_n = 0)$.)

Lemma 1.3.21. *$\mathcal{L}$ is a clone in $C_{\mathbb{T}}$.*

Proof. The formulae-in-context are clearly $\mathbb{T}$-provably functional. Showing the axioms of a clone is equivalent to showing the satisfaction of the equations given in remark 1.3.10 in their respective canonical context for the formulae defined above (see also remark 1.3.11); but this is a direct consequence of the axioms of the theory of K-algebras.

□

$\mathcal{L}$ being a clone is just saying that for a K-algebra A in Set each A^n is an A-module. Hence the above lemma can be also regarded as just another corollary of the (classical) completeness theorem [Joh02, prop. D1.5.1].

The **clone $\mathcal{A}$ of affine combinations** in $C_{\mathbb{T}}$ is defined as follows:

- $\mathcal{A}(n) = \{\vec{x}.(1 = x_1 + \ldots + x_n)\}$ (The empty sum is defined as 0, hence $\mathcal{A}(0)$ is $\{1 = 0\}$, which is the (strict) initial object in $C_{\mathbb{T}}$. See also remark 1.3.25.)

- Let α_n be the formula $1 = x_1 + \ldots + x_n$. The morphism $*_{nm} : \mathcal{A}(n) \times \mathcal{A}(m)^n \longrightarrow \mathcal{A}(m)$ is the $\mathbb{T}$-provable equivalence class of the formula-in-context

$$\vec{\lambda}, \vec{x}^1, \ldots, \vec{x}^n, \vec{y}.(\vec{y} = \lambda_1 \vec{x}^1 + \ldots + \lambda_n \vec{x}^n) \wedge \alpha_n[\vec{\lambda}/\vec{x}] \wedge \bigwedge_{1 \leq j \leq n} \alpha_m[\vec{x}^j/\vec{x}]$$

  (Note that $\mathcal{A}(n) \times \mathcal{A}(m)^n$ is $\{\vec{\lambda}, \vec{x}^1, \ldots, \vec{x}^n.\alpha_n[\vec{\lambda}/\vec{x}] \wedge \bigwedge_{1 \leq j \leq n} \alpha_m[\vec{x}^j/\vec{x}]\}$. The above formula is just the restriction of $*_{nm}$ from $\mathcal{L}$ to $\mathcal{A}$.)

- $\pi_j^n : 1 \longrightarrow \mathcal{A}(n)$ is the $\mathbb{T}$-provable equivalence class of the formula-in-context

$$\vec{y}.(\vec{y} = (0, \ldots, 0, 1, 0, \ldots, 0))$$

  with the 1 being at the jth position.

Lemma 1.3.22. *$\mathcal{A}$ is a subclone of $\mathcal{L}$ in $C_{\mathbb{T}}$.*

Proof. Due to $\phi \vdash_{\vec{x}} \top$ for any formula-in-context ϕ, $\mathcal{A}(n)$ is a subobject of $\mathcal{L}(n)$. (The monomorphism is represented by the formula $\vec{x}', \vec{x}.(\phi \wedge (\vec{x} = \vec{x}'))$ [Joh02, lem. D1.1.4(iv)].) To show $\mathcal{A}$ a subclone of $\mathcal{L}$ we need to show that the morphisms $*_{nm}$ and π^n_j restrict to $\mathcal{A}$ accordingly. This amounts to show that the sequents

$$(\vec{y} = \lambda_1\vec{x}^1 + \ldots + \lambda_n\vec{x}^n) \wedge \alpha_n[\vec{\lambda}/\vec{x}] \wedge \bigwedge_{1\leq j\leq n} \alpha_m[\vec{x}^j/\vec{x}] \vdash_{\vec{\lambda},\vec{x}^1,\ldots,\vec{x}^n,\vec{y}} 1 = y_1 + \ldots + y_m,$$

$$\vec{y} = (0, \ldots, 0, 1, 0, \ldots, 0) \vdash_{\vec{y}} 1 = y_1 + \ldots + y_n$$

are provable in $\mathbb{T}$. The second one is obvious. In the case $n = 0$ the first sequent becomes

$$(\vec{y} = \vec{0}) \wedge (0 = 1) \vdash_{\vec{y}} 1 = y_1 + \ldots + y_m$$

which is clearly provable in $\mathbb{T}$. For $n \geq 1$ the $\mathbb{T}$-provability amounts to the easy calculation

$$\sum_{j=1}^{m} y_j = \sum_{j=1}^{m}\sum_{i=1}^{n} \lambda_i x^i_j = \sum_{i=1}^{n} \lambda_i (\sum_{j=1}^{m} x^i_j) = \sum_{i=1}^{n} \lambda_i = 1$$

The $\mathbb{T}$-provability of the second sequent necessary for the functionality of $*_{nm}$ of $\mathcal{A}$ is a consequence of the $\mathbb{T}$-provability of the respective sequent of $*_{nm}$ of $\mathcal{L}$ and the Frobenius axiom. □

The syntactic category $C_{\mathbb{T}}$ contains a universal $\mathbb{T}$-model (i.e. a K-algebra) $M_{\mathbb{T}}$. The Σ-structure $M_{\mathbb{T}}$ assigns $A = \{x.\top\}$ to the sort A, and maps a function symbol $f \in \Sigma$-Fun of arity n to the formula-in-context $\vec{x}, y.(f(x_1, \ldots, x_n) = y)$. Any (cartesian) formula-in-context $\vec{x}.\phi$ has an interpretation as the subobject $\{\vec{x}.\phi\}$ of $A^n = \{\vec{x}.\top\}$. For any other subobject $\{\vec{x}.\psi\}$ we have that $\{\vec{x}.\phi\} \leq \{\vec{x}.\psi\}$, if and only if $\phi \vdash_{\vec{x}} \psi$ is provable in $\mathbb{T}$ [Joh02, lem. D1.4.5]. $M_{\mathbb{T}}$ thus satifies the axioms of a K-algebra by construction.

For any finite-limit category C and any K-algebra R in C there is a finite-limit preserving functor $F_R : C_{\mathbb{T}} \longrightarrow C$ mapping A^n to R^n and $\{\vec{x}.\phi\}$ to its interpretation $[\![\vec{x}.\phi]\!]_R$ in C. In particular, it maps the universal K-algebra $M_{\mathbb{T}}$ to R. F_R is unique up to (unique) isomorphism. In fact, this correspondence induces an equivalence of categories of the full subcategory of $[C_{\mathbb{T}}, C]$ of finite-limit preserving functors and the category $\mathbb{T}$-Mod(C) of K-algebras in C [Joh02, thm. D1.4.7]. Since clones are mapped to clones by finite-product preserving functors (proposition 1.3.3), any K-algebra in C induces a clone $\mathcal{L}_R$ of linear combinations and a clone $\mathcal{A}_R$ of affine combinations in C.

Corollary 1.3.23. *Let C be a category with finite limits. Any K-algebra R in C induces a clone $\mathcal{L}_R$ of R-linear combinations in C with $\mathcal{L}_R(n) = R^n$, and a subclone of R-affine combinations $\mathcal{A}_R$ with $\mathcal{A}_R(n)$ being the equaliser of the operation of the sum of n-elements and the constant operation* 1.

Remark 1.3.24 (*The opposite of the category of finitely presented K-algebras*)**.** Let $K\text{-Alg}_{fp}$ denote the category of finitely presented K-algebras. The category $K\text{-Alg}_{fp}^{op}$ has finite limits and a K-algebra object $K[X]$ that shares the same universal property as $M_{\mathbb{T}}$ does in $C_{\mathbb{T}}$. The functor $F_{K[X]} : C_{\mathbb{T}} \to K\text{-Alg}_{fp}^{op}$ is thus (part of) an equivalence of categories. (See, for example, [Joh02, prop. D3.1.2] and [Cos76, III.c)].) This equivalence can be also seen as follows: $F_{K[X]}$ restricts to an isomorphism of the full subcategory $\mathring{C}_{\mathbb{T}}$ generated by the objects corresponding to the Horn formulae-in-context and $K\text{-Alg}_{fp}^{op}$. Horn formulae mean finite conjunctions of atomic formulae, and the latter mean equations between terms and $\top$. The objects corresponding to cartesian formuale not contained in $\mathring{C}_{\mathbb{T}}$ are the ones involving the existential quantifier. However, by [Joh02, lemma D1.4.4(ii)] these are isomorphic to objects in $\mathring{C}_{\mathbb{T}}$.

In terms of finitely presented K-algebras the clones $\mathcal{L}$ and $\mathcal{A}$ become the respective dual structures: $\mathcal{L}(n) = K[X_1, \ldots, X_n]$, $\mathcal{A}(n) = K[X_1, \ldots, X_n]/(\sum_{i=1}^n X_i - 1)$, and the inclusion is the canonical quotient map $K[X_1, \ldots, X_n] \to K[X_1, \ldots, X_n]/(\sum_{i=1}^n X_i - 1)$. The projection π_j^n becomes the K-algebra homomorphisms $K[X_1, \ldots, X_n] \to K$ that maps X_j to 1 and the other generators X_i to 0. The morphism $*_{nm}$ becomes the K-algebra homomorphism

$$K[Y_1, \ldots, Y_m] \to K[\lambda_1, \ldots, \lambda_n] \otimes_K K[\vec{X}_1, \ldots, \vec{X}_n], \quad Y_i \mapsto \sum_{j=1}^n \lambda_j \otimes_K X_{j,i}$$

Since the morphisms in $K\text{-Alg}_{fp}^{op}$ are defined by K-algebra homomorphisms they are easier to work with than the morphisms in $\mathring{C}_{\mathbb{T}}$, which are essentially functional relations. We will make use of this fact later.

Remark 1.3.25. $\mathcal{A}(0)$ is $\{0 = 1\}$, which is strictly initial in $C_{\mathbb{T}}$. (This is easily seen from the equivalence $C_{\mathbb{T}} \simeq K\text{-Alg}_{fp}^{op}$, since $\{0 = 1\}$ corresponds to the trivial K-algebra.) In $C_{\mathbb{T}}$ this means that $\mathcal{A}$, which is representing the theory of affine spaces over K-algebras, has no constants. This is what we would expect. However, finite limit-preserving functors $C_{\mathbb{T}} \to C$ will map $\{0 = 1\}$ only to a subterminal object (the equaliser of the constants 0 and 1), which might not be an initial object, if C has one.

In the case of Set, $\mathcal{A}(0)$ is mapped to the empty set for any non-trivial K-algebra. For the trivial K-algebra it is mapped to the terminal object. So in Set we recover that the clone of affine combinations over a non-trivial K-algebra has no constants as to be expected.

For a K-algebra object R in a finite-limit category C, an affine space A over R (i.e. an $\mathcal{A}_R$-algebra) has a distinguished subobject $F_R(\{0=1\}) \times A \rightarrowtail A$. In a distributive category C this subobject is just the initial object for a non-trivial K-algebra R (i.e. $F_R(\{0=1\})$ is initial). In our applications to SDG we will be mostly working in Grothendieck toposes with local ring objects R (which are defined as being non-trivial), so we can be sure that $\mathcal{A}_R(0)$ will be initial.

1.4 Comparisons

In the last section of this chapter we shall prove the comparison theorems comparing the clones in Set with the syntactic approach, as well as clones in a complete and cocomplete cartesian closed category $\mathcal{V}$ with Lawvere theories enriched in $\mathcal{V}$. This shall make precise in which sense the approach via clones is equivalent to the other two and also justify why we may consider a clone in a category C an algebraic theory in C.

1.4.1 Clones and the syntactic approach

Our comparison theorem of clones with the syntactic approach is a generalisation of [Gou08, lem. 1.5.1 and lem. 1.5.2]. Since we have not introduced a category of presentations of algebraic theories, we shall only compare the categories of models with the categories of algebras and the clones.

(There is a notion of interpretation of a presentation of an algebraic theory in another presentation of possibly different signature [Tay93, def. 5, p.517], as well as a corresponding notion of equivalence [Tay93, def. 5, p.517]. [Tay93, thm. 4, p.517] shows that the presentations are equivalent if and only if the categories of models in Set are isomorphic as categories over Set.)

Theorem 1.4.1 (Clones and algebraic theories)**.** *Let C be a category with finite products. Clones in* Set *and algebraic theories are equivalent in the following sense*:

(1) *For every algebraic theory $\mathbb{T}$ there is a* Set-*clone $O_{\mathbb{T}}$ such that* $\mathbb{T}$-Mod(C) *and* $O_{\mathbb{T}}$-Alg(C) *are isomorphic categories over C.*

(2) *For every* Set-*clone O there is an algebraic theory $\mathbb{T}_O$ such that* $\mathbb{T}_O$-Mod(C) *and* O-Alg(C) *are isomorphic categories over C.*

(3) *The clones $O_{\mathbb{T}_O}$ and O are isomorphic. In particular, we have an isomorphism of* $O_{\mathbb{T}_O}$-Mod(C) *and* O-Alg(C) *of categories over C.*

Proof. (1) This has been shown in proposition 1.3.18 already.

(2) We define $\mathbb{T}_O$ as follows. The signature Σ has one sort A, no relation symbols and for every $n \in \mathbb{N}$ and $\sigma \in O(n)$ an n-ary function symbol $\sigma : A \cdots A \to A$. For every $n, m \in \mathbb{N}$, $\sigma \in O(n)$ and $t_1, \ldots, t_n \in O(m)$ we have a sequent

$$\top \vdash_{x_1,\ldots,x_m} \sigma(t_1(x_1, \ldots, x_m), \ldots, t_n(x_1, \ldots, x_m)) = \sigma *_{nm} (t_1, \ldots, t_n)(x_1, \ldots, x_m) \tag{1.1}$$

(In the cases $m = 0$ or $n = 0$ this is supposed to be read as $\top \vdash_{[]} \sigma(t_1, \ldots, t_n) = \sigma *_{n0} (t_1, \ldots, t_n)$ or $\top \vdash_{x_1,\ldots,x_m} \sigma = *_{0m}(\sigma)(x_1, \ldots, x_m)$, respectively. Due to normalisation we can omit the case $n = m = 0$.) For every $n \geq 1$ and $1 \leq j \leq n$ we have a sequent

$$\top \vdash_{x_1,\ldots,x_n} \pi^n_j(x_1, \ldots, x_n) = x_j \tag{1.2}$$

We define $\mathbb{T}_O$ as the set of all the sequents of type (1.1) and (1.2). Let A be a C-object. Giving a Σ-structure on A is equivalent to giving a family of maps $f_n : O(n) \to \mathrm{End}(A)(n)$, $n \in \mathbb{N}$. We may conclude from the substitution lemma 1.1.15(1) that a Σ-structure A satisfies the two sequents above, if and only if

$$f_n(\sigma) \circ (f_m(t_1), \ldots, f_m(t_n)) = f_n(\sigma *_{nm} (t_1, \ldots, t_n))$$

and $f_n(\pi^n_j) = \mathrm{pr}_j$. By remark 1.3.10 and the definition of the clone structure on $\mathrm{End}(A)$ (see example 1.3.14(a)) saying that the Σ-structure A is a $\mathbb{T}_O$-model is equivalent to saying that the family of maps f_n constitute a clone homomorphism. Clearly, a C-morphism $h : A \to A'$ is an O-algebra homomorphism if and only if it is a Σ-homomorphism. This establishes the asserted isomorphism of categories over C.

(3) We recall that $O_{\mathbb{T}_O}(n)$ is the quotient of the term algebra $F_{\mathbb{T}}(n) = T_\Sigma(n)/E_n$, where Σ is the signature from (2). (See section 1.1.4 for the respective definitions of $T_\Sigma(n)$ and E_n.)
There is a family of canonical injective maps $\iota_n : O(n) \to T_\Sigma(n)$ mapping σ to the term $\sigma(x_1, \ldots, x_n)$. We compose the ι_n with the quotient maps $q_n : T_\Sigma(n) \to F_{\mathbb{T}}(n)$. The sequents of type (1.1) and (1.2) then say that the family of maps $f_n = q_n \iota_n$ constitute a clone homomorphism $f : O \to O_{\mathbb{T}_O}$. The maps f_n are injective.
For the construction of the inverse map, note that each $O(n)$ is a $\mathbb{T}_O$-model. Indeed, $O(n)$ carries a natural Σ-structure mapping the function symbol σ to $\sigma *_{mn} (-)$, and from this construction together with the substitution lemma 1.1.15(1) it is clear that $O(n)$ satisfies all the sequents of type (1.1) (due to associativity) and (1.2) (due to the projection axiom). By proposition 1.1.21 mapping $[x_j]$ to π^n_j extends to a unique Σ-homomorphism $h_n : F_{\mathbb{T}}(n) \to O(n)$.

The $f_n : O(n) \to F_{\mathbb{T}}(n)$ are also Σ-homomorphisms that map the π_j^n to $[x_j]$. By the universal property of $F_{\mathbb{T}}(n)$ the composite $f_n \circ h_n$ must be the identity map. This shows each f_n surjective and hence f an isomorphism of clones. The asserted isomorphism of categories over C is then either a consequence of proposition 1.3.7 or can be obtained from combining the isomorphisms in (1) and (2).

□

Another proof for (3) is to combine (1) and (2) to obtain an isomorphism $O_{\mathbb{T}_O}$ -Alg$(C) \cong O$ -Alg(C) of categories over C. In particular, this holds for $C = \mathrm{Set}$. By proposition 1.3.12 $O(n)$ and $O_{\mathbb{T}_O}(n)$ are free O-algebras (respectively, free $O_{\mathbb{T}_O}$-algebras). There are bijections $h_n : O(n) \to O_{\mathbb{T}_O}(n)$ which are simultaneously isomorphisms of O and $O_{\mathbb{T}_O}$-algebras. One then has to show that the family h_n is a clone homomorphisms.

Remark 1.4.2 (*Theory of clone algebras*)**.** Theorem 1.4.1 only compares Set-clones with algebraic theories. For a syntactic approach to clones and their algebras in a finite-product category C we need to extend the many-sorted algebraic theory of clones in remark 1.3.11 to a many-sorted algebraic **theory $\mathbb{C}$ of clone algebras** over a signature Σ, which is defined as follows:

- Σ has a sort A and a sort $O(n)$ for each $n \in \mathbb{N}$.
- Σ has a function symbol

  $$*_{nm} : O(n)O(m) \cdots O(m) \to O(m)$$

  of arity $n + 1$ for every pair $(n, m) \in \mathbb{N}^2$ with $n \geq 1$, a unary function symbol

  $$*_{0m} : O(0) \to O(m)$$

  for each $m \in \mathbb{N}$, constants

  $$\pi_j^n : \to O(n)$$

  for each $n \in \mathbb{N}$ and $1 \leq j \leq n$, and an $(n + 1)$-ary function symbol

  $$\bullet_n : O(n)A \cdots A \to A$$

  for each $n \in \mathbb{N}$.
- Σ has no relation symbols

The axioms of $\mathbb{C}$ are as follows: (We employ the vector notation for lists to make the equations shorter, when necessary.)

(1) (**Clone**)

- *Associativity*: For every $n, m, k \in \mathbb{N}$

$$\top \ \vdash_{\sigma,\vec{t},\vec{s}} \ \sigma *_{nk} (t_1 *_{mk} \vec{s}, \ldots, t_n *_{mk} \vec{s}) = (\sigma *_{nm} \vec{t}) *_{mk} \vec{s}$$

  with $\sigma : O(n)$, $t_i : O(m)$, $1 \leq i \leq n$ and $s_i : O(k)$, $1 \leq i \leq m$. The cases $m = 0$ and $n = 0$ are to be interpreted as in remark 1.3.10.

- *Projection*: For every $n \geq 1$, $1 \leq j \leq n$

$$\top \ \vdash_{\vec{t}} \ \pi^n_j *_{nm} (t_1, \ldots, t_n) = t_j$$

- *Unit*: For each $n \geq 1$

$$\top \ \vdash_{\sigma:O(n)} \ \sigma *_{nn} (\pi^n_1, \ldots, \pi^n_n) = \sigma$$

- *Normalisation*: $\top \ \vdash_{t:O(0)} \ *_{00}(t) = t.$

(2) (**Associativity**) For every $n, m \in \mathbb{N}$

$$\top \ \vdash_{\sigma,\vec{t},\vec{x}} \ \sigma \bullet_n (t_1 \bullet_m \vec{x}, \ldots, t_n \bullet_m \vec{x}) = (\sigma *_{nm} \vec{t}) \bullet_m \vec{x}$$

with $\sigma : O(n)$, $t_i : O(m)$, $1 \leq i \leq n$ and $x_i : A$, $1 \leq i \leq m$. The cases $m = 0$ and $n = 0$ are to be interpreted in a similar way as the respective cases of associativity in remark 1.3.10.

(3) (**Projection**) For every $n \geq 1$, $1 \leq j \leq n$

$$\top \ \vdash_{\vec{x}} \ \pi^n_j \bullet_n (x_1, \ldots, x_n) = x_j$$

Clearly, a $\mathbb{C}$-model in C is a clone O and an O-algebra A. Conversely, a clone O in C together with an O-algebra yields a $\mathbb{C}$-model in C. A Σ-homomorphism, however, is a pair (f, h) of a clone homomorphism $f : O \to O'$ and a morphism $h : A \to A'$ such that

$$\begin{tikzcd} O(n) \times A^n \arrow[r, "f_n \times h^n"] \arrow[d, "\bullet_n"] & O'(n) \times (A')^n \arrow[d, "\bullet'_n"] \\ A \arrow[r, "h"] & A' \end{tikzcd}$$

The category $\mathbb{C}$-Mod(C) is the category of all clones and their algebras in C. For each C-clone O we have a fully faithful embedding of categories over C

$$\begin{tikzcd} O\text{-Alg}(C) \arrow[rr, tail] \arrow[dr, "U"'] & & \mathbb{C}\text{-Mod}(C) \arrow[dl, "U_C"] \\ & C & \end{tikzcd}$$

which maps each O-algebra homomorphism h to the Σ-homomorphism $(1_O, h)$.

1.4.2 Clones and enriched Lawvere theories

We turn to the comparison theorem of clones and enriched Lawvere theories. The equivalence of clones in Set and Lawvere theories has been presumably proved first in [Tay73, appendix]. A proof which shows the equivalence between the corresponding categories of models is given in [Gou08, chap. 1.5]. Since clones and Lawvere theories are conceptually very close to each other some authors were lead to consider abstract clones as categories similar to Lawvere theories (see, for example, [Tay93] or [Man76, def. 2.7]) We generalise the Set-case to cartesian closed categories.

Theorem 1.4.3 (Clones and Lawvere theories)**.** *Let $\mathcal{V}$ be a complete and cocomplete cartesian closed category.*

(1) *We can assign a clone O_L in $\mathcal{V}$ to any $\mathcal{V}$-Lawvere theory (L, α) as follows:*[7]

- $O_L(n) = L(n, 1)$
- *The morphisms* $*_{nm} : L(n, 1) \times L(m, 1)^n \to L(n, 1)$ *are given by (multi)composition*

$$L(n, 1) \times L(m, 1)^n \xrightarrow{1_{L[n,1]} \times \iota} L(n, 1) \times L(m, n) \xrightarrow{\circ} L(m, 1)$$

[7]We remind the reader of the convention that we write n for the objects $\alpha(n)$ of L.

where $\iota : L(m, 1)^n \cong L(m, n)$ *is a* $\mathcal{V}$*-natural isomorphism* $($*in* $m)$ *and '*$\circ$*' denotes the internal composition.*

- $\pi_j^n : 1 \to L(n.1)$ *are the n product projections* $n \to 1$ *in the category* L_0.

We obtain a functor $Cl : \mathrm{Law}(\mathcal{V}) \to \mathrm{clone}(\mathcal{V})$ *such that the categories* $L\text{-}\mathrm{Mod}(\mathcal{V})_0$ *and* $O_L\text{-}\mathrm{Alg}(\mathcal{V})$ *are equivalent as categories over* $\mathcal{V}$.

(2) *To any clone* O *in* $\mathcal{V}$ *we can assign a* $\mathcal{V}$*-enriched Lawvere theory* L_O *as follows*:

- *The set of objects of* L_O *is* $\mathbb{N}$.
- $L_O(n, m) = O(n)^m$
- *Composition* $L_O(k, m) \times L_O(n, k) \to L_O(n, m)$ *is the morphism*

$$O(k)^m \times O(n)^k \xrightarrow{1_{O(k)} \times \Delta_m} O(k)^m \times (O(n)^k)^m \cong (O(k) \times O(n)^k)^m \xrightarrow{(*_{kn})^m} O(n)^m$$

- *The identity morphisms are* $1 \xrightarrow{(\pi_1^n, \ldots, \pi_n^n)} L_O(n, n)$

We obtain a functor $\mathcal{L} : \mathrm{clone}(\mathcal{V}) \to \mathrm{Law}(\mathcal{V})$ *such that the categories* $L_O\text{-}\mathrm{Mod}(\mathcal{V})_0$ *and* $O\text{-}\mathrm{Alg}(\mathcal{V})$ *are equivalent as categories over* $\mathcal{V}$.

(3) Cl *and* $\mathcal{L}$ *form an equivalence of categories.*

Proof. (1) The associativity axiom for O_L follows from the coherence of the associativity diagrams of the composition in the $\mathcal{V}$-category $\mathcal{A}$ and the $\mathcal{V}$-naturality of the isomorphisms $L(m, 1)^n \cong L(m, n)$. The latter isomorphism is a consequence of the fact that the object n in L is the n-fold $\mathcal{V}$-product of 1 in L and lemma 1.2.1.

The projection and unit axioms for O_L follow from the fact that n is an n-fold product in the underlying category L_0, and that the π_j^n are the product projections for n in the underlying category L_0. The normalisation axiom is a consequence of the definition of $*_{00}$, for 0 is the $\mathcal{V}$-terminal object.

Let $F : (L, \alpha) \to (L', \alpha')$ be a morphism of $\mathcal{V}$-Lawvere theories. Since $F\alpha = \alpha'$ we have $F(n) = n$, and thus $F_{n,1} : O_L(n) \to O_{L'}(n)$. Since F preserves $\mathcal{V}$-products the underlying functor $F_0 : L_0 \to L'_0$ preserves products. In particular, F commutes with the projections, which is precisely axiom (2) in definition 1.3.2. Also, by lemma 1.2.1, as a $\mathcal{V}$-product-preserving

$\mathcal{V}$-functor, F renders the subsequent diagram commutative

$$\begin{tikzcd}
L(m,1)^n \arrow[r, "\cong"] \arrow[d, "(F_{m,1})^n"] & L(m,n) \arrow[d, "F_{m,n}"] \\
L'(m,1)^n \arrow[r, "\cong"] & L'(m,n)
\end{tikzcd}$$

Since a $\mathcal{V}$-functor commutes with composition in L and L', we see that axiom (1) in definition 1.3.2 holds. This shows that F induces a clone homomorphism $O_L \to O_{L'}$. Conversely, a clone homomorphism $f : O_L \to O_{L'}$ induces a $\mathcal{V}$-product preserving $\mathcal{V}$-functor $F : L \to L'$ by setting $F(n) = n$ and

$$F_{m,n} : L(m,n) \cong L(m,1)^n \xrightarrow{(f_m)^n} L'(m,1)^n \cong L'(m,n)$$

Indeed, since f is a clone homomorphism, F commutes with composition and units. By definition of F we have $L'(m, F(n)) \cong L'(m, F(1))^n$, and this isomorphism is $\mathcal{V}$-natural. Moreover, F commutes with the projections, since f does, and, by construction, renders commutative the respective diagram in lemma 1.2.1 showing it a $\mathcal{V}$-product preserving functor. Since F commutes with the projections, it also commutes with α and α'. F is thus a morphism of the $\mathcal{V}$-Lawvere theories $(L, \alpha) \to (L', \alpha')$. Altogether this yields a functor $Cl : \mathrm{Law}(\mathcal{V}) \to \mathrm{clone}(\mathcal{V})$, which is fully faithful.

An L-model $M : L \to \mathcal{V}$ yields a morphism $M_{n,1} : O_L(n) \to [M(n), M(1)] \cong [M(1)^n, M(1)]$ for each $n \in \mathbb{N}$. The last isomorphism is due to M preserving $\mathcal{V}$-products, and hence $M(n) \cong M(1)^n$ in $\mathcal{V}$. By a similar reasoning as for morphisms of $\mathcal{V}$-Lawvere theories we can see that this family of morphisms constitutes a clone homomorphism $O_L \to \underline{\mathrm{End}}(M(1))$. This shows $M(1)$ an O_L-algebra. (Cf. remark 1.3.15) Conversely, and in the same manner as for morphisms of $\mathcal{V}$-Lawvere theories, a clone homomorphism $f : O_L \to \underline{\mathrm{End}}(A)$ induces a $\mathcal{V}$-product preserving $\mathcal{V}$-functor $F : L \to \mathcal{V}$ such that $F(1) = A$ and $F(n) = A^n$. Due to M preserving $\mathcal{V}$-products, the $\mathcal{V}$-functor induced by the clone homomorphism $O_L \to \underline{\mathrm{End}}(M(1))$, which itself is induced by M, is $\mathcal{V}$-natural isomorphic to M. Iterating the construction while starting with a clone homomorphism $f : O_L \to \underline{\mathrm{End}}(A)$ yields f again. Moreover, to give a $\mathcal{V}$-natural transformation $h : M \to M'$ is the same as to give an O_L-algebra homomorphism $h : M(1) \to M'(1)$ in $\mathcal{V}$. (This follows from taking the exponential transpose of the commutative diagram in definition 1.3.6

and [Bor94a, prop. 6.2.8].) Altogether this yields an equivalence of categories over $\mathcal{V}$:

$$\begin{tikzcd} L\text{-Mod}(\mathcal{V})_0 \arrow[rr, "\simeq"] \arrow[dr, "\mathrm{ev}_1"'] & & O_L\text{-Alg}(\mathcal{V}) \arrow[dl, "U"] \\ & \mathcal{V} & \end{tikzcd}$$

(2) The associativity and unit axiom for O imply that L_O is indeed a $\mathcal{V}$-category. By definition of L_O each object n is the n-fold power of 1, for $L(m,n) = O(m)^n = L(m,1)^n$, which is clearly $\mathcal{V}$-natural in m. This shows that L_O has finite $\mathcal{V}$-products. Mapping each map $f : m \to n$ of finite sets m and n to the morphism $(\pi^n_{f(1)}, \ldots, \pi^n_{f(m)}) : 1 \to O(n)^m$ induces a $\mathcal{V}$-functor $\alpha_O : \mathrm{Fin}^{op} \to L_O$, which is the identity on objects. This functor preserves $\mathcal{V}$-products by construction. The pair (L_O, α_O) is thus a $\mathcal{V}$-Lawvere theory.
Let $f : O \to O'$ be a clone homomorphism. The morphisms $(f_n)^m : L_O(n,m) \to L_{O'}(n,m)$ induce a $\mathcal{V}$-functor F, which is the identity on objects. Indeed, since each f_n commutes with the projections, F commutes with the units. Since the f_n commute with the clone compositions, F commutes with the compositions in L_O and $L_{O'}$. By construction F satisfies the conditions in lemma 1.2.1, and thus preserves $\mathcal{V}$-products. It is also easy to see that F commutes with α_O and $\alpha_{O'}$, which is again due to f_n commuting with the projections. Conversely, every morphism of $\mathcal{V}$-Lawvere theories $F : (L_O, \alpha_O) \to (L_{O'}, \alpha_{O'})$ must satisfy $F_{n,m} = F^m_{n,1}$, since it preserves $\mathcal{V}$-products and we have $L(m,n) = L(m,1)^n$ and $L'(m,n) = L(m,1)^n$. We have seen already that the family of morphisms $F_{n,1} : O(n) \to O(n')$ is a clone homomorphism. This shows the existence of a fully faithful functor $\mathcal{L} : \mathrm{clone}(\mathcal{V}) \to \mathrm{Law}(\mathcal{V})$.
As regards the equivalence of L_O-models and O-algebras, this follows as in part (1) as well. Let (A, α) be an O-algebra. We know from remark 1.3.15 that α can be understood as a clone homomorphism $f : O \to \underline{\mathrm{End}}(A)$. We define a $\mathcal{V}$-functor $F : L_O \to \mathcal{V}$ as follows. Set $F(n) = A^n$ and

$$F_{n,m} : O(n)^m \xrightarrow{(f_n)^m} [A^n, A]^m \cong [A^n, A^m]$$

Since f commutes with the projections, $F_{n,n}$ commutes with the units, and since f commutes with the clone compositions, F commutes with the compositions in L_O and $\mathcal{V}$. This shows F a $\mathcal{V}$-functor. It preserves $\mathcal{V}$-products by construction. Conversely, an L_O-model $M : L_O \to \mathcal{V}$ induces a clone homomorphism $O \to \underline{\mathrm{End}}(M(1))$ by $M_{n,1} : O(n) \to [M(n), M(1)] \cong [M(1)^n, M(1)]$, as we have seen in part (1). Starting from the clone homomorphism $f : O \to \underline{\mathrm{End}}(A)$, then constructing the L_O-model, and then a clone homomorphism out of it, we obtain f again. If we iterate the construction starting from an L_O-model M, we obtain an L_O-model which is isomorphic to M. Exactly as in part (1) we can see that an O-algebra homomorphism

$h : A \to A'$ in $\mathcal{V}$ is the same as to a $\mathcal{V}$-natural transformation of the corresponding L_O-models. Altogether this yields an equivalence of categories over $\mathcal{V}$:

$$\begin{tikzcd}
O\text{-Alg}(\mathcal{V}) \arrow[rr, "\simeq"] \arrow[dr, "U"'] & & L_O\text{-Mod}(\mathcal{V})_0 \arrow[dl, "\mathrm{ev}_1"] \\
& \mathcal{V} &
\end{tikzcd}$$

(3) For the $\mathcal{V}$-Lawvere theory L_{O_L} we find that $L_{O_L}(m, n) = L(m, 1)^n$. The $\mathcal{V}$-natural isomorphism $L(m, 1)^n \cong L(m, n)$ then induces an isomorphism of $\mathcal{V}$-Lawvere theories $(L, \alpha) \cong (L_{O_L}, \alpha_{O_L})$, which is, moreover, natural in L. Conversely, we find $O_{L_O} = O$. Since both functors $\mathcal{L}$ and Cl are fully faithful, they are an equivalence of categories, as asserted.

$\square$

Chapter 2

Infinitesimal Models of Algebraic Theories

In this chapter we shall introduce and study the notion of an **infinitesimal model** of a (finitary, one-sorted) algebraic theory (IMAT). Based on the first and third approach to algebraic theories in the first chapter we introduce and compare two viewpoints on IMATs:

(1) *Internal structures*: IMATs are infinitesimal structures together with a right action of a clone in finite-product categories. We call them *infinitesimal algebras*.

(2) *Cartesian logic*: IMATs are models of a cartesian theory associated to an algebraic theory $\mathbb{T}$ in finite-limit categories. We call it the *infinitesimalisation* of $\mathbb{T}$.

A possible third viewpoint is to consider IMATs in a complete and cocomplete cartesian category $\mathcal{V}$ as $\mathcal{V}$-product and monomorphism-preserving $\mathcal{V}$-functors from a $\mathcal{V}$-category L constructed out of a $\mathcal{V}$-Lawvere theory L. This point of view might be interesting, if one is interested to study generalisations of IMATs to a monoidal setting. However, since there are currently no applications of this viewpoint, we have omitted it.

We begin this chapter by introducing the notion of *infinitesimal structure*, which is an abstraction from the notion of infinitesimal simplices in the theory of formal manifolds in *Synthetic Differential Geometry* (SDG). The main and natural example of infinitesimal structures that we will continuously develop in this and the next chapter are the *nil-square infinitesimal structures*. The nil-square infinitesimal structures are of key importance for applications. For example, the infinitesimal simplices in SDG constitute a nil-square infinitesimal structure.

Next we introduce *infinitesimal algebras* for a clone O and generalise some of the basic results in chapter 1.3. We show that every object in the syntactic category of the algebraic theory

$\mathbb{T}$ of (commutative, unital) K-algebras corresponding to a Horn formula is an infinitesimal algebra of the clone of affine combinations over the nil-square infinitesimal structure, and that every morphism becomes a homomorphism of these algebras. An equivalent result has been obtained by Kock in [Koc15] for K-algebras. Our approach differs from Kock in that we give a (largely) synthetic proof using the syntactic category and categorical logic instead of the category of K-algebras. Furthermore, we also clarify the result by employing the respective structures, which are not available in [Koc15]. If we translate this approach into the language of K-algebras this means we are working with presentations of K-algebras rather than the K-algebra structure itself, as it is done in [Koc15]. The syntactic approach is more susceptible to generalisations. Three such generalisations will be presented. First, we extend the result from Horn formulae to certain cartesian formulae involving the existential quantifier, and in chapter 3 we extend the result to the syntactic category of C^∞*-rings* and *well-adapted models*.

In view of the comparison theorem of clones with the syntactic approach in the first chapter, it is natural to ask what the corresponding notion of an IMAT would be in the syntactic approach to algebraic theories. For this purpose we introduce the construction of *infinitesimalisation*. For algebraic theories we show that, in a finite limit category, the models of the infinitesimalisation of $\mathbb{T}$ are the same thing as infinitesimal $O_{\mathbb{T}}$-algebras. To be able to extend this result to clones O in a finite limit category C we introduce an infinitesimalisation of cartesian theories. Unlike the infinitesimalisation of algebraic theories this construction turns out to be too general for our purposes, however; we use it only as a guide. We then define a theory of infinitesimal affine spaces and infinitesimal clone algebras and give the respective comparison theorems with the clone approach.

It turns out that the infinitesimalisation can be carried out for any first-order theory. This allows us, in principle, to define what it means for a space to be *infinitesimally projective*, or *infinitesimally euclidean* (using Tarski's first-order axiomatisation of Euclidean geometry, for example). Unfortunately, we are not able to answer these questions in this thesis. They are subject of future research.

In the last section of this chapter we study the properties of categories of IMAT's over a Grothendieck topos $\mathcal{S}$. We show that the category of infinitesimal O-algebras is locally presentable, that the forgetful functor to $\mathcal{S}$ has a left adjoint, lifts limits, lifts pushouts of spans of *infinitesimal structure reflecting morphisms*, reflects regular epimorphisms, and that it is a regular category. For infinitesimally affine spaces, in particular, we obtain also that the forgetful functor lifts colimits of diagrams of infinitesimal structure reflecting morphisms. An application of this fact is that formal manifolds in SDG are infinitesimally affine spaces, as we will discuss in chapter 3.2.2.

2.1 Infinitesimal structures

We want to define IMATs as certain partial algebras of clones. The family of subobjects the clone will act on will be given by the *infinitesimal structure*. This structure is an abstraction of the family of infinitesimal simplices of a formal manifold in SDG [Koc06, chap. I.18], [Koc09, chap. 2.1], whence the name.

Definition 2.1.1 (*Infinitesimal structure*)**.** Let C be a category with finite products. An **infinitesimal structure** over A in C (or *i-structure* for short) is a subobject $A\langle n\rangle \rightarrowtail A^n$ for each $n \in \mathbb{N}$, such that

(1) $A\langle 1\rangle = A$

(2) $A\langle 0\rangle \cong 1$

(3) Let $n \geq 1$ and let $\mathrm{pr}_{j_k} : A^n \to A$ be a family of projections onto the j_kth factor for $1 \leq k \leq m$. The restriction of the morphism $(\mathrm{pr}_{j_1}, \dots, \mathrm{pr}_{j_m})$ to $A\langle n\rangle$ factors through $A\langle m\rangle$:

$$\begin{tikzcd} A\langle n\rangle \arrow[r, tail] \arrow[d] & A^n \arrow[d, "(\mathrm{pr}_{j_1},\dots,\mathrm{pr}_{j_m})"] \\ A\langle m\rangle \arrow[r, tail] & A^m \end{tikzcd}$$

In other words, an i-structure over A is a subobject $A\langle -\rangle$ of the functor $n \mapsto A^n$ in $[\mathrm{Fin}^{op}, C]$ satisfying (1) and (2). A **homomorphism of infinitesimal structures** (or *i-morphism* for short) is a C-morphism $f : A \to A'$ such that each f^n restricts to $A\langle n\rangle \to A'\langle n\rangle$

$$\begin{tikzcd} A\langle n\rangle \arrow[r, tail] \arrow[d] & A^n \arrow[d, "f^n"] \\ A'\langle n\rangle \arrow[r, tail] & (A')^n \end{tikzcd}$$

We denote the restriction by f^n again. With the identities and composition taken from C, i-structures and i-morphisms form a category I-Str(C) over C. (We could have defined I-Str(C) also as the full subcategory of $[\mathrm{Fin}^{op}, C]$ generated by the i-structures. This would yield the same category.)

The intuition is that $A\langle n\rangle$ is the set of n-tuples of points that are mutual *infinitesimal neighbours*, and hence infinitesimally close to each other. I-morphisms are maps that preserve the infinitesimal neighbourhoods. However, as it is the case with many structures that are abstractions of particular examples to general structures, the original intuition is easily jeopardised. This is also the case for infinitesimal structures as exhibited by the right adjoint functor in the subsequent proposition.

Proposition 2.1.2. *The forgetful functor $U : \text{I-Str}(C) \to C$ has a left adjoint Δ, and a right adjoint T. Δ maps A to the* **discrete i-structure** *Δ_A defined by $\Delta_A\langle n\rangle = \Delta_n$ for $n \geq 1$ and $\Delta_A\langle 0\rangle = 1$, with $\Delta_n : A \rightarrowtail A^n$ being the diagonal map. T maps A to the indiscrete, or* **total i-structure** *with $T(A)\langle n\rangle = A^n$.*

Proof. Δ_A and $T(A)$ are clearly i-structures and the mappings $A \mapsto \Delta_A$ and $A \mapsto T(A)$ together with the identity map on morphisms are clearly functorial. The asserted adjunctions follow from the evident natural bijections $\text{I-Str}(C)(\Delta_A, B\langle -\rangle) \cong C(A, B)$ and $\text{I-Str}(C)(A\langle -\rangle, T(B)) \cong C(A, B)$. □

Sometimes it is convenient to speak of just an i-structure A without specifying the base object $A\langle 1\rangle$. Clearly, any subfunctor A of $T(B)$ satisfying (2) and (3) in definition 2.1.1 is an i-structure over $A\langle 1\rangle$. This follows from axiom (3) for the case $m = 1$.

2.1.1 Nil-square infinitesimal structures

Let $\mathbb{T}$ be the algebraic theory of (commutative, unital) K-algebras and $C_{\mathbb{T}}$ its syntactic category. We use the notation introduced in the example of clones of linear and affine combinations in the previous chapter: A is the unique sort and A^n stands for $\{\vec{x}.\top\}$, where n is the length of the context $\vec{x}$. The **nil-square i-structure** over A^n is defined as follows. Let ∂_n be the formula $\bigwedge_{1\leq i,j\leq n}(x_i - y_i)(x_j - y_j) = 0$. Set $A^n\langle 0\rangle = 1$, $A^n\langle 1\rangle = A^n$ and

$$A^n\langle m\rangle = \{\vec{x}^1, \dots, \vec{x}^m. \bigwedge_{1\leq k,\ell\leq m} \partial_n[\vec{x}^k, \vec{x}^\ell/\vec{x}, \vec{y}]\}$$

Projections $\text{pr}_j : A^n\langle m\rangle \to A^n$ are represented by the formulae-in-context

$$\vec{x}^1, \dots, \vec{x}^m, \vec{z}.(\vec{z} = \vec{x}^j) \wedge \bigwedge_{1\leq k,\ell\leq m} \partial_n[\vec{x}^k, \vec{x}^\ell/\vec{x}, \vec{y}]$$

and the morphism representing $(\text{pr}_{j_1}, \dots, \text{pr}_{j_r})$ is just the conjunction of the formulae representing the pr_{j_i} (cf., for example, [Joh02, lem. D1.4.2]). By the second identity rule we have

$$\bigwedge_{1\leq k\leq r} (\vec{z}^k = \vec{x}^{j_k}) \wedge \bigwedge_{1\leq k,\ell\leq m} \partial_n[\vec{x}^k, \vec{x}^\ell/\vec{x}, \vec{y}] \vdash_{\vec{x}^1,\dots,\vec{x}^m,\vec{z}^1,\dots,\vec{z}^r} \bigwedge_{1\leq k,\ell\leq r} \partial_n[\vec{z}^k, \vec{z}^\ell/\vec{x}, \vec{y}]$$

This shows (3) in definition 2.1.1, and hence that the $A^n\langle n\rangle$ do form an i-structure over A^n.

The intuition behind the nil-square i-structure is as follows. The formula ∂_n defines a symmetric and reflexive relation '$\sim_1$' on A^n; this is $A\langle 2\rangle$. We say that P and Q are (first-order) *i-neighbours*, if $P \sim_1 Q$. An n-tuple of points $(P_1, \ldots, P_m)$ lies in $A^n\langle m\rangle$, if and only if the m points are mutual i-neighbours, i.e. $P_i \sim_1 P_j$ for $1 \le i, j \le m$. If $P \sim_1 Q$ then any quadratic form maps $P - Q$ to 0. In particular, any (Riemannian) distance of P and Q is 0. P and Q can thus be considered as being infinitesimally close to each other.

Nil-square i-structures can be defined over any object $\{\vec{x}.\phi\}$ in $C_{\mathbb{T}}$ simply by forming the conjuction of the defining formula and ϕ for each factor:

$$\{\vec{x}.\phi\}\langle m\rangle = \{\vec{x}^{\,1}, \ldots, \vec{x}^{\,m}. \bigwedge_{1\le k,\ell\le m} \partial_n[\vec{x}^{\,k}, \vec{x}^{\,\ell}/\vec{x}, \vec{y}] \wedge \bigwedge_{1\le k\le m} \phi[\vec{x}^{\,k}/\vec{x}]\}$$

The inclusion $i : \{\vec{x}.\phi\} \rightarrowtail A^n$ becomes then an i-morphism. In categorical terms this construction is the pullback of the subobject $A^n\langle -\rangle \rightarrowtail T(A^n)$ along $T(i) : T(\{\vec{x}.\phi\}) \longrightarrow T(A^n)$ in $[\mathrm{Fin}^{op}, C_{\mathbb{T}}]$. Note that for *closed formulae*, i.e. where the context is empty, the nil-square i-structure becomes total.

For every functor $F : C \longrightarrow D$ that preserves finite products and monomorphisms the induced functor by post-composition $F_* : [\mathrm{Fin}^{op}, C] \longrightarrow [\mathrm{Fin}^{op}, D]$ restricts to a functor $F_* : \text{I-Str}(C) \longrightarrow \text{I-Str}(D)$. Due to the universal property of $C_{\mathbb{T}}$ all the 'spaces' in a finite-limit category C, which can be constructed out of a K-algebra object R by means of a cartesian formula-in-context $\vec{x}.\phi$ (over a signature of K-algebras), come equipped with a canonical nil-square i-structure. Such nil-square i-structures are also the infinitesimal structures used in SDG, where they are called *infinitesimal simplices* [1].

Our aim is to show that the morphisms in $C_{\mathbb{T}}$ become i-morphisms of the nil-square i-structures. The syntactic definition of morphisms in $C_{\mathbb{T}}$ is cumbersome, however. In remark 1.3.24 we remarked that $C_{\mathbb{T}}$ is equivalent to $K\text{-Alg}_{fp}^{op}$, the opposite category of finitely presented K-algebras, and the latter is isomorphic to the category $\mathring{C}_{\mathbb{T}}$, the full subcategory of $C_{\mathbb{T}}$ generated by the objects corresponding to Horn formulae-in-context in $\mathbb{T}$. We can thus use K-algebra homomorphisms instead of the morphisms in $C_{\mathbb{T}}$. This will give us the desired result for $\mathring{C}_{\mathbb{T}}$ and we will study how to extend it to $C_{\mathbb{T}}$ later.

[1] Note the shift in index by 1 for the nil-square i-structures when compared to infinitesimal simplices in [Koc06] and [Koc09]. What we call $M\langle 2\rangle$ would be $M_{<1>}$ there. The reason for our choice of indexing should become obvious when we introduce infinitesimal algebras in the next section.

When we translate the nil-square i-structures into the language of finitely presented algebras, then for $m \geq 2$ the object $A^n\langle m\rangle$ is mapped to the quotient map

$$K[\vec{X}_1, \ldots, \vec{X}_m] \twoheadrightarrow K[\vec{X}_1, \ldots, \vec{X}_m]/I_{\partial_{n,m}}$$

where each $\vec{X}_i = X_{i,1}, \ldots, X_{i,n}$ is a list of distinct variables and $I_{\partial_{n,m}}$ is the K-algebra ideal generated by the set

$$\{(X_{k,i} - X_{\ell,i})(X_{k,j} - X_{\ell,j}) \mid 1 \leq k, \ell \leq m,\ 1 \leq i, j \leq n\}$$

(Note that we need to keep track of the set of generators of the ideals, since we are working with *finitely presented* K-algebras, and not just *finitely presentable* ones.) For a Horn formula-in-context $\vec{x}.\phi$ the object $\{\vec{x}.\phi\}\langle m\rangle$ is mapped to the quotient map

$$K[\vec{X}_1, \ldots, \vec{X}_m]/mI \twoheadrightarrow K[\vec{X}_1, \ldots, \vec{X}_m]/(mI \cup I_{\partial_{n,m}})$$

where mI is the ideal generated by the disjoint union $\coprod_{1\leq j\leq m} I_{\phi[\vec{x}_j/\vec{x}]}$ and I_ϕ is the ideal constructed out of the formula ϕ. This can be done as follows. Terms-in-context are (provably equivalent to) polynomials in the variables of the context. An equation $s = t$ adds the polynomial $t - s$ to the generating set. Conjunctions are translated into the join of the ideals of the two subformulas.

Remark 2.1.3. As regards the existential quantifier we could attempt to perform the dual construction of the interpretation of a formula-in-context $\vec{x}.(\exists! y)\phi$ for a Horn formula ϕ in the category of finitely presented K-algebras (for example, as defined in [Joh02, def. D1.3.4f]):

$$K[\vec{X}] \xrightarrow{(-)\otimes 1} K[\vec{X}, Y] \longrightarrow\!\!\!\!\!\rightarrow K[\vec{X}, Y]/I_\phi$$

and hope to define $I_{(\exists! y)\phi}$ as the kernel of the composite homomorphism. Indeed, the composite homomorphism is epi (by soundness). However, it is, in general, not a regular epimorphism, and thus not the quotient of its kernel. For example, the closed formula $(\exists! y)2y = 1$ for $K = \mathbb{Z}$ is interpreted by the epimorphism $\mathbb{Z} \twoheadrightarrow \mathbb{Z}[2^{-1}] = \mathbb{Z}[Y]/(2Y - 1)$, which is also a monomorphism, but not an isomorphism.

We wish to show that any K-algebra homomorphism $f : K[\vec{X}]/I \to K[\vec{Y}]/J$ becomes an i-morphism of the nil-square i-structures in $K\text{-Alg}_{fp}^{op}$. Let n and k be the lengths of the lists $\vec{X}$ and $\vec{Y}$, respectively. We need to show that the homomorphism $f^{\otimes m}$ factors through the respective quotients for $m \geq 2$:

$$\begin{tikzcd}
K[\vec{X}_1, \dots, \vec{X}_m]/mI \arrow[r, two heads] \arrow[d, "f^{\otimes m}"'] & K[\vec{X}_1, \dots, \vec{X}_m]/(mI \cup I_{\partial_{n,m}}) \arrow[d, dashed] \\
K[\vec{Y}_1, \dots, \vec{Y}_m]/mJ \arrow[r, two heads] & K[\vec{Y}_1, \dots, \vec{Y}_m]/(mJ \cup I_{\partial_{k,m}})
\end{tikzcd} \tag{2.1}$$

As before we have represented the m-fold tensor products with the corresponding generators and relations. The ideals mI and mJ stand again for the ideals generated by the m-fold disjoint union of the ideal I, respectively J, while replacing each variable X_i with the corresponding $X_{j,i}$ in the jth factor I, respectively $Y_{j,i}$ for Y_i in J. (Similarly, in the ideal $I_{\partial_{k,m}}$ in the bottom row, we assume that the variables $X_{j,i}$ have been replaced by the corresponding variables $Y_{j,i}$.)

To begin with we consider a K-algebra homomorphism $f : K[\vec{X}] \to K[\vec{Y}]$. The homomorphism f is uniquely determined by the polynomials $P_i(\vec{Y}) = f(X_i)$, $1 \leq i \leq n$. To show the factorisation (2.1) in this case, it is sufficient to consider the case $m = 2$:

$$\begin{tikzcd}
K[\vec{X}_1, \vec{X}_2] \arrow[r, two heads] \arrow[d, "f \otimes_K f"'] & K[\vec{X}_1, \vec{X}_2]/I_{\partial_{n,2}} \arrow[d, dashed] \\
K[\vec{Y}_1, \vec{Y}_2] \arrow[r, two heads] & K[\vec{Y}_1, \vec{Y}_2]/I_{\partial_{k,2}}
\end{tikzcd}$$

This amounts to show that $(P_i(\vec{Y}_1) - P_i(\vec{Y}_2))(P_j(\vec{Y}_1) - P_j(\vec{Y}_2))$ lies in $I_{\partial_{n,2}}$ for $1 \leq i, j \leq n$. The following version of *Hadamard's lemma* for polynomials implies that this is indeed the case. (For future use we prove a slightly stronger result than we would need here.)

Lemma 2.1.4 (Hadamard's lemma)**.** *Let R be a commutative ring (with 1) and let P be a polynomial in $R[\vec{X}]$ with $\vec{X} = X_1, \dots, X_n$. There are n polynomials $L_i \in R[\vec{X}]$ and n^2 polynomials $Q_{ij} \in R[\vec{X}, \vec{Y}] \cong R[\vec{X}] \otimes_R R[\vec{X}]$ such that*

$$P(\vec{Y}) - P(\vec{X}) = \sum_{i=1}^{n} (Y_i - X_i) L_i(\vec{X}) + \sum_{i,j=1}^{n} (Y_i - X_i)(Y_j - X_j) Q_{ij}(\vec{X}, \vec{Y})$$

Proof. We show that there are polynomials $L_i \in R[\vec{X}]$ and $Q_{ij} \in R[\vec{X}, \vec{H}]$ such that

$$P(\vec{X} + \vec{H}) = P(\vec{X}) + \sum_{i=1}^{n} H_i L_i(\vec{X}) + \sum_{i,j=1}^{n} H_i H_j Q_{ij}(\vec{X}, \vec{H})$$

The asserted result follows from this by substituting $\vec{Y} - \vec{X}$ for $\vec{H}$. The proof is by induction on the number of variables n.

$(n = 1)$ Let $P(X) = \sum_{k=0}^{m} a_k X^k$ with $a_k \in R$. Apply the binomial formula and reorder terms

$$\begin{aligned} P(X+H) &= \sum_{k=0}^{m} a_k (X+H)^k = \sum_{k=0}^{m}\sum_{\ell=0}^{k} a_k \binom{k}{\ell} H^\ell X^{k-\ell} \\ &= \sum_{\ell=0}^{m}\Big(\sum_{k=\ell}^{m}\binom{k}{\ell} a_k X^{k-\ell}\Big) H^\ell \\ &= \sum_{k=0}^{m} a_k X^k + \Big(\sum_{k=1}^{m} k a_k X^{k-1}\Big) H + \Big(\sum_{\ell=2}^{m}\sum_{k=\ell}^{m}\binom{k}{\ell} a_k X^{k-\ell} H^{\ell-2}\Big) H^2 \\ &= P(X) + L(X)H + Q(X,H)H^2 \end{aligned}$$

$(n \Rightarrow n+1)$ Firstly, apply the case $n = 1$ to the polynomial $P(\vec{Z}, X+H)$ in the ring $R[\vec{Z}][X]$:

$$P(\vec{Z}, X+H) = P(\vec{Z}, X) + L(\vec{Z}, X)H + Q(\vec{Z}, X, H)H^2$$

Substitute $\vec{X} + \vec{H}$ for $\vec{Z}$, X_{n+1} for X, and H_{n+1} for H:

$$\begin{aligned} P(X_1 + H_1, \ldots, X_{n+1} + H_{n+1}) &= P(\vec{X} + \vec{H}, X_{n+1} + H_{n+1}) \\ &= P(\vec{X} + \vec{H}, X_{n+1}) + L(\vec{X} + \vec{H}, X_{n+1})H_{n+1} \\ &\quad + Q(\vec{X} + \vec{H}, X_{n+1}, H_{n+1})H_{n+1}^2 \end{aligned}$$

Now apply the induction hypothesis to $P(\vec{X} + \vec{H}, X_{n+1})$ and $L(\vec{X} + \vec{H}, X_{n+1})$, where P and L are considered as polynomials in $R[X_{n+1}][\vec{X}]$:

$$\begin{aligned} P(\vec{X} + \vec{H}, X_{n+1} + H_{n+1}) = P(\vec{X}, X_{n+1}) &+ \sum_{i=1}^{n} H_i L_i^0(\vec{X}, X_{n+1}) + \sum_{i,j=1}^{n} H_i H_j Q_{ij}^0(\vec{X}, X_{n+1}, \vec{H}) \\ &+ L(\vec{X}, X_{n+1})H_{n+1} + \sum_{i=1}^{n} H_i H_{n+1} L_i^1(\vec{X}, X_{n+1}) \\ &+ \sum_{i,j=1}^{n} H_i H_j H_{n+1} Q_{ij}^1(\vec{X}, X_{n+1}, \vec{H}) + Q(\vec{X} + \vec{H}, X_{n+1}, H_{n+1})H_{n+1}^2 \end{aligned}$$

Define $L_i \in R[X_1, \ldots, X_{n+1}]$ as L_i^0 for $1 \leq i \leq n$ and L_{n+1} as L. Define

$$Q_{ij} \in R[X_1, \ldots, X_{n+1}, H_1, \ldots, H_{n+1}] \quad \text{as} \quad Q_{ij} = \begin{cases} Q_{ij}^0 & \text{for } 1 \leq i, j \leq n, \\ L_i^1 & \text{for } 1 \leq i \leq n \text{ and } j = n+1, \\ \sum_{k=1}^n H_k Q_{kj}^1 & \text{for } 1 \leq j \leq n \text{ and } i = n+1, \\ Q & i = j = n+1. \end{cases}$$

This gives the desired representation.

□

We come back to the general case of a K-algebra homomorphism $f : K[\vec{X}]/I \to K[\vec{Y}]/J$. By choosing representatives $P_i(\vec{Y})$ for each equivalence class $f([X_i])$ we obtain a K-algebra homomorphism $\bar{f} : K[\vec{X}] \to K[\vec{Y}]$, such that $f(X_i) = P_i(\vec{Y})$. The homomorphism $\bar{f}$ is a lift of f along the quotient maps:

$$\begin{array}{ccc} K[\vec{X}] & \twoheadrightarrow & K[\vec{X}]/I \\ {\scriptstyle \bar{f}}\downarrow & & \downarrow{\scriptstyle f} \\ K[\vec{Y}] & \twoheadrightarrow & K[\vec{Y}]/J \end{array}$$

We can apply our previous result to $\bar{f}$:

$$\begin{array}{ccc} K[\vec{X}_1, \ldots, \vec{X}_m] & \twoheadrightarrow & K[\vec{X}_1, \ldots, \vec{X}_m]/I_{\partial_{n,m}} \\ {\scriptstyle \bar{f}^{\otimes m}}\downarrow & & \downarrow \\ K[\vec{Y}_1, \ldots, \vec{Y}_m] & \twoheadrightarrow & K[\vec{Y}_1, \ldots, \vec{Y}_m]/I_{\partial_{k,m}} \end{array}$$

It says that $\bar{f}^{\otimes m}$ maps the ideal $I_{\partial_{n,m}}$ into the ideal $I_{\partial_{k,m}}$. Clearly, $\bar{f}$ maps the ideal I into J; hence $\bar{f}^{\otimes m}$ maps the ideal mI into mJ. This gives us the factorisation in (2.1) as required.

With this we can extend the previous observation: not only does each object in $C_{\mathbb{T}}$ have a nil-square i-structure, but every morphism $f : \{\vec{x}.\phi\} \to \{\vec{y}.\psi\}$ is also an i-morphism, if the formulae ϕ and ψ are Horn formulae. We have proved the following theorem:

Theorem 2.1.5. *Let $\mathring{C}_{\mathbb{T}}$ denote the full subcategory generated by the objects $\{\vec{x}.\phi\}$, where ϕ is a Horn formula. The forgetful functor U : I-Str$(C_{\mathbb{T}}) \to C_{\mathbb{T}}$ has a section $N : \mathring{C}_{\mathbb{T}} \to$ I-Str$(C_{\mathbb{T}})$ (i.e. $UN = I_{\mathring{C}_{\mathbb{T}}}$) that maps each object $\{\vec{x}.\phi\}$ to the nil-square i-structure $\{\vec{x}.\phi\}\langle - \rangle$ over $\{\vec{x}.\phi\}$.*

The subcategory $\mathring{C}_{\mathbb{T}}$ is isomorphism dense in $C_{\mathbb{T}}$. In particular, it is a category that has all finite limits. The construction of the nil-square i-structure also makes use of the Horn fragment only, so U restricts to a functor $U : \text{I-Str}(\mathring{C}_{\mathbb{T}}) \to \mathring{C}_{\mathbb{T}}$. The subsequent corollary makes precise how a K-algebra object R induces nil-square i-structures and i-morphisms on 'spaces' that can be constructed out of R syntactically.

Corollary 2.1.6. *Let C be a category with finite limits, and R a K-algebra object in C. Let $F_R : C_{\mathbb{T}} \to C$ be the functor that maps the universal K-algebra $M_{\mathbb{T}}$ to R. We have*

$$\begin{tikzcd} \text{I-Str}(\mathring{C}_{\mathbb{T}}) \arrow[r, "(F_R)_*"] & \text{I-Str}(C) \arrow[d, "U"] \\ \mathring{C}_{\mathbb{T}} \arrow[u, tail, "N"] \arrow[r, "F_R"] & C \end{tikzcd}$$

Remarks 2.1.7 (*Finitely generated algebras*)**.**

(a) The argument presented gives a stronger result. We have not needed the fact that the ideals are finitely generated. In fact, the given definition of the nil-square i-structure makes sense for any (presentation of a) finitely generated K-algebra. Our argument then shows that any K-algebra homomorphism becomes an i-morphism in the opposite category of finitely generated algebras $K\text{-Alg}^{op}_{fg}$.

If we choose a presentation for each finitely generated K-algebra, then there is a functor $N : K\text{-Alg}^{op}_{fg} \to \text{I-Str}(K\text{-Alg}^{op}_{fg})$ that maps each presentation to its nil-square i-structure, and we have $UN \cong I_{K\text{-Alg}^{op}_{fg}}$ for the forgetful functor $U : \text{I-Str}(K\text{-Alg}^{op}_{fg}) \to K\text{-Alg}^{op}_{fg}$. Any other choice of presentations will yield an isomorphic functor to N, for all morphisms in $K\text{-Alg}^{op}_{fg}$ become i-morphisms for the nil-square i-structure.

In other words, every *finitely generated affine scheme* admits a nil-square i-structure. Considering that the idea of nil-square infinitesimals and the first-order neighbourhood of the diagonal are taken from algebraic geometry, we should expect at least this much.

(b) From the viewpoint of categorical logic $K\text{-Alg}^{op}_{fg}$ is equivalent to the syntactic category of the theory $\mathbb{T}$ of K-algebras in *infinitary cartesian logic*, and isomorphic to the syntactic category of infinitary Horn logic. In this fragment of infinitary first-order logic we are allowed to form infinitary conjuctions indexed by arbitrary sets, not just finite ones, as long as the set of free variables of the conjunction remains finite. The observation that we did not need the ideals to be finitely generated becomes that we did not need the formulae-in-context to be build up from finite conjunctions only. In fact, we did not need to assume anything about the formulae-in-context whatsoever to be able to define the nil-square i-structures. In principle, we can define them on objects in the syntactic category of $\mathbb{T}$ in any fragment of infinitary first-order logic (that extends cartesian logic). However, it remains to be shown that each morphism lifts to an i-morphism. We study this for cartesian formulae-in-context that contain existential quantifiers .

The objects $\{\vec{x}.\psi\}$ that are not contained in $\mathring{C}_{\mathbb{T}}$ are precisely the ones where the corresponding cartesian formulae ψ contain an existential quantifier. We know that each object $\{\vec{x}.\psi\}$ is isomorphic to an $\{\vec{x}.\psi'\}$ such that ψ' is quantifier free ([Joh02, lem. D1.4.4(ii)]). Moreover, every such $\vec{x}.\psi$ is provably equivalent to a cartesian formula-in-context $\vec{x}.(\exists!\vec{y})\phi$, where ϕ is a Horn formula ([Joh02, lem. D1.3.8(i)]). Our aim is to show that the isomorphism

$$f : \{\vec{x},\vec{y}.\phi\} \to \{\vec{x}'.(\exists!\vec{y})\phi[\vec{x}'/\vec{x}]\}$$

represented by the formula $\vec{x},\vec{x}',\vec{y}.\phi \wedge (\vec{x} = \vec{x}')$ is an isomorphism of i-structures. (See the proof of [Joh02, lem. D1.4.4(ii)] that f is indeed an isomorphism.) Once we have established that f lifts to an isomorphism of i-structures we can extend theorem 2.1.5 and its subsequent corollary to $C_{\mathbb{T}}$ easily.

Clearly, f is an i-morphism; but it is not obvious that its inverse f^{-1}, which is represented by the same formula, is an i-morphism as well. For f^{-1} to become an i-morphism we have to show that the sequent

$$\partial_n[\vec{x}_1,\vec{x}_2/\vec{x},\vec{y}] \wedge \phi[\vec{x}_1,\vec{y}_1/\vec{x},\vec{y}] \wedge \phi[\vec{x}_2,\vec{y}_2/\vec{x},\vec{y}] \vdash_{\vec{x}_1,\vec{x}_2,\vec{y}_1,\vec{y}_2} \partial_{n+m}[(\vec{x}_1,\vec{y}_1),(\vec{x}_2,\vec{y}_2)/\vec{x},\vec{y}] \quad (2.2)$$

is $\mathbb{T}$-provable. (Here n and m denote the length of the contexts $\vec{x}$ and $\vec{y}$, respectively.) This is because the nil-square i-structure we defined on $\{\vec{x},\vec{y}.\phi\}$ depends on both the contexts $\vec{x}$ and $\vec{y}$, whereas on $\{\vec{x}.(\exists!\vec{y})\phi\}$ it only depends on $\vec{x}$. We have to show that if $\vec{x}_1$ and $\vec{x}_2$ are i-neighbours, then the uniquely determined $\vec{y}_1$ and $\vec{y}_2$ are such that the points $(\vec{x}_1,\vec{y}_1)$ and $(\vec{x}_2,\vec{y}_2)$ are i-neighbours, too.

Example 2.1.8. An example[2] that is important for applications in Synthetic Differential Geometry is the space of invertible elements R^* in a K-algebra R:

$$R^* := \{x \in R \mid (\exists!y)\, xy = 1\}$$

If we would want to define the space of invertible elements using the Horn fragment only, we would consider the space

$$U := \{(x,y) \in R^2 \mid xy = 1\}$$

[2]This example has been suggested to the author by Anders Kock.

We know that the projection onto the first component map $\mathrm{pr}_1 : R^2 \to R$ induces an isomorphism of U and R^*. (This is just the interpretation of the cartesian existential quantifier.) Clearly, pr_1 is an i-morphism. However, a priori it is not clear that its inverse $R^* \to U$ is an i-morphism as well. After all, the i-structure on U is the restriction of the nil-square i-structure on R^2 and the i-structure of R^* is the restriction of the nil-square i-structure of R. In this particular example we find that for $x_1 \sim_1 x_2$ we can set $y_2 = y_1 - (x_2 - x_1)y_1^2$, and hence conclude that $(x_1, y_1) \sim_1 (x_2, y_2)$ in R^2. Indeed, due to $(x_2 - x_1)^2 = 0$ we see that

$$\begin{aligned} x_2 y_2 &= (x_1 + (x_2 - x_1))(y_1 - (x_2 - x_1)y_1^2) \\ &= 1 - (x_2 - x_1)y_1 + (x_2 - x_1)y_1 + (x_2 - x_1)^2 y_1^2 \\ &= 1 \end{aligned}$$

and hence $(x_2, y_2) \in U$. By construction we have that $(y_2 - y_1)^2 = 0$ and $(x_2 - x_1)(y_2 - y_1) = 0$, so $(x_1, y_1) \sim_1 (x_2, y_2)$ as asserted. In fact, what we have used here is that the the equation $xy = 1$ implicitly defines the inversion map $R^* \to R$ mapping each element to its multiplicative inverse, and that this map is differentiable in the sense of SDG.

Remark 2.1.9. It is important to note that we are not trying to extend the result of theorem 2.1.5 by transporting the i-structure from objects of $\mathring{C}_{\mathbb{T}}$ to objects of $C_{\mathbb{T}}$ along isomorphisms; i.e. making use of the fact that $\mathring{C}_{\mathbb{T}}$ is an isomorphism dense subcategory of $C_{\mathbb{T}}$. Such an extension is trivially possible, since i-structures are transportable along isomorphisms. However, we have defined the i-structure on each object of $C_{\mathbb{T}}$ as the nil-square i-structure already, and the problem is to show that each morphism in $C_{\mathbb{T}}$ lifts to an i-morphism for this choice of i-structure. With theorem 2.1.5 we have shown that the choice of nil-square i-structure is natural for $\mathring{C}_{\mathbb{T}}$, and we want to show that it is also natural for $C_{\mathbb{T}}$ as well.

Let $\vec{x}.(\exists!\vec{y})\phi$ be a cartesian formula-in context with ϕ being a Horn formula. Without loss of generality we can assume that ϕ is the conjunction of $\ell \geq 1$ polynomial equations of the form

$$\phi \equiv (P_1(\vec{x}, \vec{y}) = 0) \wedge \ldots \wedge (P_\ell(\vec{x}, \vec{y}) = 0) \tag{2.3}$$

(All the polynomials have coefficients in K.) Let n be the length of the context $\vec{x}$ and m be the length of the context $\vec{y}$. We shall think of ϕ as a system of polynomial equations that define a function $g : \{\vec{x}.(\exists!\vec{y})\phi\} \to A^n$ implicitly by $\vec{y} = g(\vec{x})$. This makes $\{\vec{x}, \vec{y}.\phi\}$ the graph of the function g. First we show that the system of polynomial equations is overdetermined at worst, i.e. $\ell \geq m$.

Lemma 2.1.10. *In the system of polynomial equations in (2.3) that are $\mathbb{T}$-provably functional in $\vec{x}$ we have $\ell \geq m$.*

Proof. In this and the subsequent proofs we shall use a formal notation and arguments as if we were doing formal analysis of rational functions between algebraic sets. (In fact, a large part of our argument is basically just an application the Kock-Lawvere axiom in the classifying topos Set[$\mathbb{T}$] of K-algebras.) We define a function

$$F(\vec{x},\vec{y}) = \begin{pmatrix} P_1(\vec{x},\vec{y}) \\ \vdots \\ P_\ell(\vec{x},\vec{y}) \end{pmatrix}$$

With this notation we have that ϕ is equivalent to $F(\vec{x},\vec{y}) = 0$. Let $\vec{d} \in A^m$ (this means it is a context of the same length as $\vec{y}$) be a i-neighbour of 0, i.e. $d_i d_j = 0$, $1 \leq i, j \leq m$, and consider $F(\vec{x},\vec{y}+\vec{d}) - F(\vec{x},\vec{y})$. We apply Hadamard's lemma (lemma 2.1.4) to each component F_j of F, and collect the polynomials L_{ij} in an $\ell \times m$ matrix $\partial_{\vec{y}} F(\vec{x},\vec{y})$. This yields

$$F(\vec{x},\vec{y}+\vec{d}) - F(\vec{x},\vec{y}) = \partial_{\vec{y}} F(\vec{x},\vec{y})[\vec{d}]$$

We want to show that for $\vec{x} \in A^n$ and $\vec{y} \in A^m$ such that $F(\vec{x},\vec{y}) = 0$ the kernel of $\partial_{\vec{y}} F(\vec{x},\vec{y})$ is trivial. In other words, we want to show the sequent

$$(F(\vec{x},\vec{y}) = 0) \wedge (\partial_{\vec{y}} F(\vec{x},\vec{y})[\vec{h}] = 0) \vdash_{\vec{x},\vec{y},\vec{h}} \vec{h} = 0$$

provable in $\mathbb{T}$. Let $\vec{h} \in A^m$ be such that $\partial_{\vec{y}} F(\vec{x},\vec{y})[\vec{h}] = 0$. For a $d \in A$ with $d^2 = 0$ we have that $d\vec{h}$ is an i-neighbour of 0, and hence

$$F(\vec{x},\vec{y}+d\vec{h}) = \partial_{\vec{y}} F(\vec{x},\vec{y})[d\vec{h}] = d\partial_{\vec{y}} F(\vec{x},\vec{y})[\vec{h}] = 0$$

Since $F(\vec{x},\vec{y}) = 0$ implies that the $\vec{y}$ is uniquely determined by $\vec{x}$ we find $d\vec{h} = 0$. To show that $\vec{h} = 0$ we consider the situation in K-Alg_{fp}^{op}. Let $R = K[\vec{x},\vec{y},\vec{h}]/I$ where I is the ideal generated by the all the equations of $F(\vec{x},\vec{y}) = 0$ and $\partial_{\vec{y}} F(\vec{x},\vec{y})[\vec{h}] = 0$. We consider $R[d]/(d^2)$. By the previous argument we obtain $dh_j = 0$ for $1 \leq j \leq m$ in $R[d]/(d^2)$. Since every element of $R[d]/(d^2)$ is of the form $a + bd$ with uniquely determined $a, b \in R$, $dh_j = 0$ implies $h_j = 0$ in R for each j. This shows that $\vec{h} = 0$, respectively that the asserted sequence is provable in $\mathbb{T}$. Let $R = K[\vec{x},\vec{y}]/I_\phi$ where I_ϕ is the ideal generated by the all the equations of $F(\vec{x},\vec{y}) = 0$. The matrix $\partial_{\vec{y}} F(\vec{x},\vec{y})$ induces an R-linear map $R^m \to R^\ell$. Due to the choice of R the kernel of this

R-linear map is trivial. By a well-known result of commutative algebra we can conclude that $m \leq \ell$. □

For the next step we consider the case $m = \ell$ first, and show that the sequent (2.2) is $\mathbb{T}$-provable in this case. A similar type of argument as in the proof of the preceding lemma can be employed. In the preceding proof we have shown that for $\vec{x} \in A^n$ and $\vec{y} \in A^m$ such that $F(\vec{x}, \vec{y}) = 0$ the kernel of $\partial_{\vec{y}} F(\vec{x}, \vec{y})$ is trivial. According to a result of linear algebra over commutative rings the determinant of the matrix $\partial_{\vec{y}} F(\vec{x}, \vec{y})$ is not a zero divisor [McD84, cor. I.30]. We show that it must be invertible. To see this we consider once again the K-algebra $K[\vec{x}, \vec{y}, \vec{h}]/I_\phi$, where I_ϕ is the ideal generated by the all the equations of $F(\vec{x}, \vec{y}) = 0$, and form the quotient with the ideal generated by $\det \partial_{\vec{y}} F(\vec{x}, \vec{y})$. Let us denote the resulting K-algebra by R. The matrix $\partial_{\vec{y}} F(\vec{x}, \vec{y})$ induces an R-linear map $R^\ell \longrightarrow R^\ell$, and this map has to be injective. On the other hand, by construction of R the determinant of the linear map is 0 in R. This can only be the case, if R is the trivial K-algebra; but then the ideal generated by $\det \partial_{\vec{y}} F(\vec{x}, \vec{y})$ must be R and $\det \partial_{\vec{y}} F(\vec{x}, \vec{y})$ is invertible in R. From this we may conclude that the sequent

$$F(\vec{x}, \vec{y}) = 0 \vdash_{\vec{x}, \vec{y}} (\exists! z)\, z \det \partial_{\vec{y}} F(\vec{x}, \vec{y}) = 1$$

is provable in $\mathbb{T}$. Since the determinant is invertible, we can construct an inverse for the matrix $\partial_{\vec{y}} F(\vec{x}, \vec{y})$ from its adjugate matrix.

In the next step we shall apply an infinitesimal version of the implicit function theorem. Let $\vec{x}_1 \sim_1 \vec{x}_2$ and suppose $F(\vec{x}_1, \vec{y}_1) = 0$. We set

$$\vec{y}_2 = \vec{y}_1 - [\partial_{\vec{y}} F(\vec{x}, \vec{y})]^{-1} \partial_{\vec{x}} F(\vec{x}, \vec{y})[\vec{x}_2 - \vec{x}_1]$$

(Here $\partial_{\vec{x}} F(\vec{x}, \vec{y})$ denotes the matrix obtained from Hadamard's lemma by varying the $\vec{x}$ variable, but keeping $\vec{y}$ fixed.) From the definition of $\vec{y}_2$ it follows by a straight-forward calculation that $(\vec{x}_1, \vec{y}_1) \sim_1 (\vec{x}_2, \vec{y}_2)$. Applying Hadamard's lemma thus yields

$$F(\vec{x}_2, \vec{y}_2) - F(\vec{x}_1, \vec{y}_1) = \partial_{\vec{x}} F(\vec{x}, \vec{y})[\vec{x}_2 - \vec{x}_1] + \partial_{\vec{y}} F(\vec{x}, \vec{y})[\vec{y}_2 - \vec{y}_1]$$

By construction of $\vec{y}_2$ we see that the right hand side of the equation evaluates to 0. Since we have assumed $F(\vec{x}_1, \vec{y}_1) = 0$ we obtain $F(\vec{x}_2, \vec{y}_2) = 0$. Moreover, since $\vec{y}_2$ is uniquely determined by $\vec{x}_2$ we conclude the sequent (2.2). As the nil-square i-structure is defined by forming conjuctions of the neighbourhood relation '$\sim_1$', i.e. $(\vec{x}^1, \ldots, \vec{x}^m)$ are i-neighbours in $\{\vec{x}.(\exists! \vec{y})\phi\}$ (that is 'lie' in $\{\vec{x}.(\exists! \vec{y})\phi\}\langle m\rangle$) if and only if $\vec{x}^k \sim_1 \vec{x}^\ell$, $1 \leq k, \ell \leq m$, we have shown:

Theorem 2.1.11. *Let $\vec{x}.(\exists!\vec{y})\phi$ be a cartesian formula-in context with ϕ being a conjunction of $\ell \geq 1$ polynomial equations of the form $(P_1(\vec{x},\vec{y}) = 0) \wedge \ldots \wedge (P_\ell(\vec{x},\vec{y}) = 0)$. If ℓ is the length of the context $\vec{y}$ then the isomorphism*

$$f : \{\vec{x},\vec{y}.\phi\} \to \{\vec{x}'.(\exists!\vec{y})\phi[\vec{x}'/\vec{x}]\}$$

represented by the formula $\vec{x},\vec{x}',\vec{y}.\phi \wedge (\vec{x} = \vec{x}')$ is an isomorphism of the respective nil-square i-structures.

For the case that there are more polynomial equations than variables in $\vec{y}$, i.e. the system of polynomial equations is overdetermined, we cannot apply the implicit function theorem argument anymore. Yet the intuition remains that if the system has a solution (which it does by assumption), then the variables in $\vec{y}$ should be rational functions of $\vec{x}$. This intuition can be made precise with the help of a result from computational commutative algebra known as the Shape Lemma. To be able to apply this lemma we will have to restrict ourselves to K-algebras where K is an integral domain of characteristic zero. Even though we cannot provide a proof for the general case here, K being an integral domain of characteristic zero does cover the two most important cases for applications, namely rings ($K = \mathbb{Z}$) and K being the field of rationals.

Theorem 2.1.12. *Let K be an integral domain of characteristic 0 and $\mathbb{T}$ the cartesian theory of K-algebras. Let $\vec{x}.(\exists!\vec{y})\phi$ be a cartesian formula-in context with ϕ being a conjunction of $\ell \geq 1$ polynomial equations of the form $(P_1(\vec{x},\vec{y}) = 0) \wedge \ldots \wedge (P_\ell(\vec{x},\vec{y}) = 0)$. If the ideal $I_\phi \hookrightarrow K[\vec{x},\vec{y}]$ generated by the polynomials P_j satisfies $I_\phi \cap K[\vec{x}] = \{0\}$ then the isomorphism*

$$f : \{\vec{x},\vec{y}.\phi\} \to \{\vec{x}'.(\exists!\vec{y})\phi[\vec{x}'/\vec{x}]\}$$

represented by the formula $\vec{x},\vec{x}',\vec{y}.\phi \wedge (\vec{x} = \vec{x}')$ is an isomorphism of the respective nil-square i-structures.

Proof. (i) We begin by showing that under the given conditions, the components of the solution $\vec{y}$ of the system of polynomial equations defined by ϕ are rational functions of $\vec{x}$. K is an integral domain and hence so is $K[\vec{x}]$. We can thus construct its quotient field, which is the field of rational functions $K(\vec{x})$ with coefficients in K. There is a natural embedding of K-algebras $K[\vec{x},\vec{y}] \hookrightarrow K(\vec{x})[\vec{y}]$ and we consider the ideal $I \leq K(\vec{x})[\vec{y}]$ generated by the image of I_ϕ under this embedding. Due to $I_\phi \cap K[\vec{x}] = \{0\}$ the ideal I is not trivial. By Hilbert's Nullstellensatz its corresponding zero set in $\overline{K(\vec{x})}^m$ is not-empty. (Here $\overline{K(\vec{x})}$ denotes the algebraic closure of $K(\vec{x})$ and m is the length of the context $\vec{y}$.)

Since

$$\phi \wedge \phi[\vec{x}, \vec{y}'/\vec{x}, \vec{y}] \vdash_{\vec{x},\vec{y},\vec{y}'} \vec{y} = \vec{y}'$$

is provable in $\mathbb{T}$ the zero set of I contains exactly one point, and the ideal in $\overline{K(\vec{x})}$ generated by I is maximal. From this it follows that I is a zero-dimensional radical ideal to which we can apply the Shape Lemma [GM87, prop. 1.6]. The latter states that there is a Gröbner basis for I of the form

$$(y_1 - g_1(y_m), \ldots, y_{m-1} - g_{m-1}(y_m), g_m(y_m))$$

where $g_j \in K(\vec{x})[y_m]$, the degree of g_m equals the number of distinct zeros of I and $\deg g_j < \deg g_m$ for $j < m$. Since the number of distinct zeros is one in our case, we obtain that there a rational functions $r_j \in K(\vec{x})$ such that $I = (y_1 - r_1, \ldots, y_m - r_m)$. This shows that the solution y_j of the system of polynomial equations are rational functions of $\vec{x}$ as asserted.

(ii) Let $r_j = f_j/g_j$ with $f_j, g_j \in K[\vec{x}]$ for $1 \leq j \leq m$. In step (i) we have shown that the sequent and its converse

$$\phi \dashv\vdash_{\vec{x},\vec{y}} (\exists! \vec{z})(y_1 = f_1(\vec{x})z_1) \wedge \ldots \wedge (y_m = f_m(\vec{x})z_m) \wedge (g_1(\vec{x})z_1 = 1) \wedge (g_m(\vec{x})z_m = 1)$$

are provable in $\mathbb{T}$. We can now employ an argument like in example 2.1.8 to show that for $\vec{x}_1 \sim_1 \vec{x}_2$ and the solution $\vec{y}_1$ to the system of polynomial equations for the parameters $\vec{x}_1$ we can construct a solution $\vec{y}_2$ for the parameter $\vec{x}_1$ such that $(\vec{x}_1, \vec{y}_1) \sim_1 (\vec{x}_2, \vec{y}_2)$. By calculating the formal derivative of each r_j (either by using the chain rule, or by evaluating $r_j(\vec{x}_1 + (\vec{x}_2 - \vec{x}_1))$ directly) we know that the variables in $\vec{y}_2$ have to satisfy

$$y_{2,j} = y_{1,j} + (g_j(\vec{x}_1)\partial f_j(\vec{x}_1)[\vec{x}_2 - \vec{x}_1] - f_j(\vec{x}_1)\partial g_j(\vec{x}_1)[\vec{x}_2 - \vec{x}_1])z_j^2$$

On the other hand we can use this to define $\vec{y}_2$. It is then a a straight-forward calculation to check that $(\vec{x}_1, \vec{y}_1) \sim_1 (\vec{x}_2, \vec{y}_2)$. We conclude that f^{-1} preserves i-structure and hence that f is an isomorphism of i-structures as asserted.

□

The condition on the ideal I_ϕ is saying that we cannot eliminate the $\vec{y}$-dependence using algebraic operations with the polynomials P_j from the formula ϕ. This seems a rather artificial requirement and we expect that a more profound analysis of this problem should allow us to drop it. If that is the case, then we can truly extend theorem 2.1.5 to the whole of $C_{\mathbb{T}}$ for the case of K being an integral domain of characteristic 0. Since we are lacking a proof, however, we can only state it as a conjecture.

Conjecture Let K be an integral domain of characteristic 0 and $\mathbb{T}$ the theory of K-algebras. The forgetful functor $U : \text{I-Str}(C_{\mathbb{T}}) \to C_{\mathbb{T}}$ has a section $N : C_{\mathbb{T}} \to \text{I-Str}(C_{\mathbb{T}})$ (i.e. $UN = I_{C_{\mathbb{T}}}$) that maps each object $\{\vec{x}.\phi\}$ to the nil-square i-structure $\{\vec{x}.\phi\}\langle - \rangle$ over $\{\vec{x}.\phi\}$. In this sense the nil-square i-structure becomes a natural choice on $C_{\mathbb{T}}$.

2.1.2 Infinitesimal structures generated by relations

I-structures generated by relations are best understood when formulated in cartesian logic. To be able to apply cartesian logic to a finite-limit category C we need to use the *internal language* of C. We begin by recalling the definition from [Joh02, chap. D1.3].

Definition 2.1.13 (*Internal language*)**.** Let C be a small category with finite limits. Define the *canonical signature* Σ_C of C as follows:

- Σ_C-Sort = ob C, i.e. for each object A in C we have a sort $\ulcorner A \urcorner$.
- Σ_C-Fun has one function symbol

$$\ulcorner f \urcorner : \ulcorner A_1 \urcorner \cdots \ulcorner A_n \urcorner \to \ulcorner B \urcorner$$

  for each morphism $f : A_1 \times \ldots \times A_n \to B$.
- Σ_C-Rel has one relation symbol

$$\ulcorner R \urcorner \rightarrowtail \ulcorner A_1 \urcorner \cdots \ulcorner A_n \urcorner$$

  for every subobject $R \rightarrowtail A_1 \times \ldots \times A_n$.

The first-order language over this signature is called the **internal language** of C.

C carries a natural Σ_C-structure mapping $\ulcorner A \urcorner$ to A, $\ulcorner f \urcorner$ to f and $\ulcorner R \urcorner$ to R. We shall say that C *satisfies a sequent* σ (or that σ is valid in C), and write $C \models \sigma$, if this Σ_C-structure satisfies σ. The fragment of first-order logic (cartesian, regular, coherent, etc.) that can be interpreted in C depends on the properties of the category C. For now we will only use cartesian logic.

The internal language allows us to work in and reason about C as if the objects were sets and the morphisms were maps. We can define (and construct) objects via formulae-in-context and prove things about C using a (suitable) deduction system. For example, due to its soundness the (cartesian) sequents that are satisfied in C are closed under the rules of inference in the deduction system for cartesian logic introduced in the first chapter.

We shall follow the common abuse of notation and do not distinguish between objects A, morphisms f and relations R, and their respective names $\ulcorner A \urcorner$, $\ulcorner f \urcorner$ and $\ulcorner R \urcorner$. This leads to ambiguities like that we may consider $A_1 \times A_2$ as either the product of sorts A_1 and A_2 in C, or the sort $A_1 \times A_2$. Similarly, $f : A_1 \times A_2 \longrightarrow B$ may be considered as a binary function symbol, or a unary one. Such ambiguities should not cause any trouble in practice; in fact, they may be welcome.

Furthermore, we shall use the internal language in categories C that are not small. To make this work with our set-based definitions one could always just restrict the signature to some small subcategory closed under finite limits containing the objects and morphisms we are reasoning about (respectively be a regular, coherent, etc. such subcategory). However, a more common and less cumbersome practice is to simply extend the definition of signature to allow proper classes, and hence obtain proper classes of terms and formulae as it is done in [Joh02, chap. D1.3] and [MM92, chap. IV.5]. We shall follow this practice here as well and tacitly extend the set-based definitions to allow proper classes whenever it is needed.

We rephrase the definition of an i-structure in terms of the internal language.

Lemma 2.1.14. *Let C be a category with finite limits. A family of subobjects $A\langle n\rangle \rightarrowtail A^n$ for $n \in \mathbb{N}$ is an i-structure over A, if and only if the following sequents are satisfied in C:*

(1) $\top \vdash_{x:A} A\langle 1\rangle(x)$

(2) $\top \vdash_{[]} A\langle 0\rangle$

(3) *For every map $f : m \longrightarrow n$ of finite sets*

$$A\langle n\rangle(x_1, \ldots, x_n) \;\vdash_{x_1:A,\ldots,x_n:A}\; A\langle m\rangle(x_{f(1)}, \ldots, x_{f(m)})$$

Proof. (1) This sequent is satisfied, if and only if the maximal subobject A is a subobject of $A\langle 1\rangle$, i.e., if and only if $A\langle 1\rangle = A$. (Strictly speaking this just shows the subobjects to be equal and hence the monomorphism $A\langle 1\rangle \rightarrowtail A$ an isomorphism. However, in this case we will always choose A as the representative of the subobject.)

(2) The equivalence is due to a similar argument as given in (1).

(3) The interpretation of the atomic formula-in-context $\vec{x}.A\langle m\rangle(x_{f(1)}, \ldots, x_{f(m)})$ is the pullback

$$\begin{tikzcd}
[\![\vec{x}.A\langle m\rangle(x_{f(1)}, \ldots, x_{f(m)})]\!] \arrow[r] \arrow[d, tail] & A\langle m\rangle \arrow[d, tail] \\
A^n \arrow[r, "(\mathrm{pr}_{f(1)},\ldots,\mathrm{pr}_{f(m)})"] & A^m
\end{tikzcd}$$

The sequent is satisfied in C if and only if $A\langle n\rangle$ factors through $[\![\vec{x}.A\langle m\rangle(x_{f(1)}, \ldots, x_{f(m)})]\!]$; but this is the case if and only if (3) in definition 2.1.1 holds for $j_i = f(i)$, $1 \leq i \leq m$.

$\square$

In the same vein a morphism $f : A \to A'$ is an i-morphism if and only if

$$A\langle n\rangle(x_1, \ldots, x_n) \vdash_{\vec{x}} A'\langle n\rangle(f(x_1), \ldots, f(x_n))$$

is satisfied in C for all $n \geq 2$.

A nil-square i-structure A in $C_{\mathbb{T}}$ has the property that it is *generated* by the relation $A\langle 2\rangle$. We have made use of this fact in the proof of theorem 2.1.5. If we view the $A\langle n\rangle$ as n-ary relations then 'being generated by $A\langle 2\rangle$' means in the internal language of the category $C_{\mathbb{T}}$ that the sequent

$$\bigwedge_{1\leq i,j\leq n} A\langle 2\rangle(x_i, x_j) \vdash_{x_1,\ldots,x_n} A\langle n\rangle(x_1, \ldots, x_n)$$

is satisfied in $C_{\mathbb{T}}$. In other words, $(P_1, \ldots, P_n)$ is in $A\langle n\rangle$, if all pairs (P_i, P_j) are in $A\langle 2\rangle$. A glance at the definition of the nil-square i-structure shows that this is exactly how it was defined. (In more geometric terms we would say that a nil-square i-structure is generated by the first neighbourhood of the diagonal.)

Now let C be a category with finite limits and A an i-structure in C. Clearly, lemma 2.1.14(3) implies that each $A\langle n\rangle$ is a symmetric and reflexive (n-ary) relation. Conversely, let R be a symmetric and reflexive binary relation over an object A. Using the internal language of C we can define an i-structure $A[R]$ over A generated by R. We set $A[R]\langle 0\rangle$ to 1 and define $A[R]\langle n\rangle$ for $n \geq 1$ as the C-interpretations

$$[\![\vec{x}. \bigwedge_{1\leq i,j\leq n} R(x_i, x_j)]\!] \rightarrowtail A^n$$

For $n = 1$ this is $[\![x.R(x, x)]\!] \rightarrowtail A$. Since R is reflexive, i.e. the sequent $\top \vdash_{x:A} R(x, x)$ is satisfied in C, we can take A to be the interpretation of $[\![x.R(x, x)]\!]$. This gives $A[R]\langle 1\rangle = A$.

We need to show that the sequents in lemma 2.1.14(3) are satisfied. Let $f : m \to n$ be a map of non-empty finite sets. $A[R]\langle m\rangle(x_{f(1)}, \dots, x_{f(m)})$ is the interpretation $[\![\vec{x}.\ \bigwedge_{1\leq i,j\leq m} R(x_{f(i)}, x_{f(j)})]\!]$. The sequent

$$\bigwedge_{1\leq i,j\leq n} R(x_i, x_j) \vdash_{\vec{x}} \bigwedge_{1\leq i,j\leq m} R(x_{f(i)}, x_{f(j)})$$

is clearly satisfied in C, so $A[R]$ is an i-structure as asserted. Moreover, since R is symmetric and reflexive we have $R \cong A[R]\langle 2\rangle$. We say that an i-structure A **is generated by** R, if $A[R]$ and A are isomorphic as i-structures. Note that for any i-structure A we always have that A is an i-substructure of $A[A\langle 2\rangle]$. This is a consequence of the interpretation of sequents and lemma 2.1.14(3), for we can introduce the conjunction over all maps $f : 2 \to n$

$$A\langle n\rangle(x_1, \dots, x_n) \vdash_{\vec{x}} \bigwedge_{1\leq i,j\leq n} A\langle 2\rangle(x_i, x_j)$$

Let $(a, b) : R \rightarrowtail A \times A$. In categorical terms the construction of $A[R]\langle n\rangle$ amounts to taking the pullback of the finite family of n^2 subobjects $R \times A^{n-2} \rightarrowtail A^n$, where the monomorphisms are obtained from the various combinations of projections and the morphisms $a : R \to A$ and $b : R \to A$.

We can define $A[R]$ for arbitrary relations over A by the above formulae. However, if R fails to be reflexive, then $A[R]$ is not an i-structure over A, but only over the subobject $[\![x.R(x, x)]\!] \rightarrowtail A$. If R fails to be symmetric then $A[R]\langle 2\rangle \rightarrowtail R$ is a proper subobject only; namely the symmetric part of R.

2.2 Infinitesimal O-algebras

We are now ready to define infinitesimal algebras of a clone O in a finite-product category C. As mentioned before O will act on the infinitesimal structure instead of the finite powers of A. The axioms are essentially the same as for O-algebras. However, to make associativity work we need to introduce a further axiom, the *neighbourhood axiom*.

Definition 2.2.1 (*Infinitesimal O-algebra*)**.** Let C be category with finite products and O a clone in C. An **infinitesimal O-algebra** (A, α) (or *i-O-algebra* for short) is an i-structure A in C together with a family of morphisms

$$\alpha_n : O(n) \times A\langle n\rangle \to A\langle 1\rangle, \qquad n \in \mathbb{N}$$

such that the subsequent diagrams are rendered commutative:

(1) (**Neighbourhood**) For every pair $(n, m) \in \mathbb{N}^2$ we have a factorisation as indicated by the dashed arrow in the subsequent diagram:

$$\begin{tikzcd}
& O(m)^n \times (A\langle m\rangle)^n \arrow[dr, "\cong"] & \\
O(m)^n \times A\langle m\rangle \arrow[ur, "1_{O(m)^n}\times\Delta_{A\langle m\rangle}"] \arrow[d, dashed] & & (O(m) \times A\langle m\rangle)^n \arrow[d, "(\alpha_m)^n"] \\
A\langle n\rangle \arrow[rr, tail] & & A\langle 1\rangle^n
\end{tikzcd}$$

where $\Delta_{A\langle m\rangle} : A\langle m\rangle \longrightarrow (A\langle m\rangle)^n$ is the diagonal map. We will denote the dashed morphism by $(\alpha_m)^n$ again. (Note that due to $A\langle 0\rangle \cong 1$ the case $n = 0$ is trivial.)

(2) (**Associativity**) For every pair $(n, m) \in \mathbb{N}^2$

$$\begin{tikzcd}
& O(n) \times O(m)^n \times A\langle m\rangle \arrow[dl, "*_{nm}\times 1_{A\langle m\rangle}"'] \arrow[dr, dashed, "1_{O(n)}\times(\alpha_m)^n"] & \\
O(m) \times A\langle m\rangle \arrow[dr, "\alpha_m"'] & & O(n) \times A\langle n\rangle \arrow[dl, "\alpha_n"] \\
& A\langle 1\rangle &
\end{tikzcd}$$

where the dashed arrow indicates the use of the factorisation in (1).

(3) (**Projection**) For every $n \geq 1$, $1 \leq j \leq n$

$$\begin{tikzcd}
1 \times A\langle n\rangle \arrow[r, "\pi_j^n\times 1_{A\langle n\rangle}"] \arrow[d, "\cong"'] & O(n) \times A\langle n\rangle \arrow[d, "\alpha_n"] \\
A\langle n\rangle \arrow[r, "\mathrm{pr}_j"] & A\langle 1\rangle
\end{tikzcd}$$

where $\mathrm{pr}_j : A\langle n\rangle \rightarrowtail (A\langle 1\rangle)^n \longrightarrow A\langle 1\rangle$ denotes the (restriction of the) projection onto the jth factor.

The intuition behind the neighbourhood axiom is that if we fix an infinitesimal neighbourhood of points $(P_1, \ldots, P_n)$ in $A\langle n\rangle$ then any operation on those points will yield a point that is infinitesimally close to these points again. In other words, the O-algebra generated by $(P_1, \ldots, P_n)$ lies in the infinitesimal neighbourhood of $(P_1, \ldots, P_n)$. Note that if O has constants then the neighbourhood axiom for $m = 0$ says that all constants are mutual infinitesimal neighbours, and lie in the infinitesimal neighbourhood of each point in $A\langle 1\rangle$.

Definition 2.2.2 (*Homomorphism of infinitesimal O-algebras*)**.** Let O be a clone in C and (A, α), (A', α') two i-O-algebras. An **infinitesimal O-algebra homomorphism** (or *i-O-homomorphism* for short) is an i-morphism $h : A \to A'$ rendering the following diagram commutative for every $n \in \mathbb{N}$

$$\begin{tikzcd} O(n) \times A\langle n\rangle \arrow[r, "1_{O(n)}\times h^n"] \arrow[d, "\alpha_n"] & O(n) \times A'\langle n\rangle \arrow[d, "\alpha'_n"] \\ A\langle 1\rangle \arrow[r, "h"] & A'\langle 1\rangle \end{tikzcd}$$

With the composition and identities taken from I-Str(C), i-O-algebras together with i-O-homomorphisms form a category O -IAlg(C) over C.

Remarks 2.2.3.

(a) Every O-algebra (A, α) induces an i-O-algebra $(T(A), \alpha)$ on the total i-structure over A, since for $T(A)$ the neighbourhood axiom is trivially satisfied. This construction yields a fully faithful embedding of O -Alg(C) into O -IAlg(C) as categories over C. Contrary to what the adjective 'infinitesimal' might suggest, infinitesimal O-algebras generalise O-algebras as a structure.

In general, we shall not distinguish between an object A and an i-structure A over the object A, unless we are dealing with multiple i-structures over A at once. If an i-O-algebra (A, α) is an O-algebra we shall sometimes stress that by saying that (A, α) is a **total O-algebra**.

(b) It might seem surprising that we did not need any of the properties of i-structures apart from it being a collection of subobjects of the finite powers of a fixed object. The properties $A\langle 0\rangle \cong 1$ and $A\langle 1\rangle = A$ are just normalisation conditions. The first condition allows us to deal with constants conveniently, the second is only because of our interest in i-structures over a particular 'space'. The third condition for i-structures is contained implicitly in definition 2.2.1, and follows from the neighbourhood and projection axioms.

I-structures may be hence considered as i-algebras for the clone of the (algebraic) theory of equality. Indeed, if C is a distributive category then we have such a clone. It is the initial clone O_I constructed from the copowers of 1 (see example 1.3.19). For any i-structure A we can define

$$\iota_n : O_I(n) \times A\langle n\rangle \to A\langle 1\rangle, \qquad n \in \mathbb{N}$$

as the unique morphism $\{\mathrm{pr}_1, \ldots, \mathrm{pr}_n\}$ given by the universal property of the coproduct, where $\mathrm{pr}_j : A\langle n\rangle \to A\langle 1\rangle$ denotes the (restriction of) the projection onto the jth factor. The neighbourhood axiom for ι is just property (3) for A in definition 2.1.1. (A, ι) is an i-O_I-algebra and the construction induces an isomorphism of O_I -IAlg(C) and I-Str(C) as categories over C. (In fact, it yields the inverse to the forgetful functor O_I -IAlg$(C) \to$ I-Str(C).)

We shall see in remark 2.2.5(c) how i-structures can be seen as i-algebras of the theory of equality in every category C with finite products. Before we can do that we need to extend remark 1.3.16 and define i-O-algebras in a finite-product category C for a Set-clone O.

Definition 2.2.4 (*i-algebras of* Set-*clones*)**.** Let C be a category with finite products, A an i-structure in C, and O a clone in Set. An *i-O-algebra* (A, α) is a family of maps

$$\alpha_n : O(n) \to C(A\langle n\rangle, A\langle 1\rangle), \qquad n \in \mathbb{N}$$

with the following properties:

(1) (**Neighbourhood**) For any pair $(n, m) \in \mathbb{N}^2$, $n \geq 1$ and $t_1, \ldots, t_n \in O(m)$ the morphism $(\alpha_m(t_1), \ldots, \alpha_m(t_n))$ factors through $A\langle n\rangle$

$$\begin{tikzcd} A\langle m\rangle \arrow[r, "(\alpha_m(t_1),\ldots,\alpha_m(t_n))"] \arrow[dr, dashed] & A^n \\ & A\langle n\rangle \arrow[u, tail] \end{tikzcd}$$

We will denote the dashed morphism by $(\alpha_m(t_1), \ldots, \alpha_m(t_n))$ again.

(2) (**Composition**) For every pair $(n, m) \in \mathbb{N}^2$, $\sigma \in O(n)$ and $t_1, \ldots, t_n \in O(m)$

$$\alpha_m(\sigma *_{nm} (t_1, \ldots, t_n)) = \alpha_n(\sigma) \circ (\alpha_m(t_1), \ldots, \alpha_m(t_n))$$

The case $n = 0$ means $\alpha_m(*_{0m}(\sigma)) = \alpha_0(\sigma) \circ !_{A\langle m\rangle}$

$$\begin{tikzcd} A\langle m\rangle \arrow[rr, "\alpha_m(*_{0m}(\sigma))"] \arrow[dr, "!_{A\langle m\rangle}"'] & & A\langle 1\rangle \\ & A\langle 0\rangle \cong 1 \arrow[ur, "\alpha_0(\sigma)"'] & \end{tikzcd}$$

(3) (**Projection**) For every $n \geq 1$ and $1 \leq j \leq n$

$$\alpha_n(\pi_j^n) = \mathrm{pr}_j$$

where $\mathrm{pr}_j : A\langle n\rangle \rightarrowtail A\langle 1\rangle^n \to A\langle 1\rangle$ is the (restriction of the) projection onto the jth factor.

An *i-O-algebra homomorphism* $h : (A, \alpha) \to (A', \alpha')$ in C is an i-morphism $h : A \to A'$ rendering commutative the diagrams

$$\begin{tikzcd}
A\langle n\rangle \arrow[r, "h^n"] \arrow[d, "\alpha_n(\sigma)"'] & A'\langle n\rangle \arrow[d, "\alpha'_n(\sigma)"] \\
A\langle 1\rangle \arrow[r, "h"] & A'\langle 1\rangle
\end{tikzcd}$$

for all $n \in \mathbb{N}$ and $\sigma \in O(n)$. Composition and identities are taken from C. We shall denote the category of i-O-algebras in C by O -IAlg(C) again. It will be clear from the context whether O is a C-clone, or a Set-clone, and hence which definition we mean.

Remarks 2.2.5.

(a) In the case that A is the total i-structure, i.e. $A \cong T(A\langle 1\rangle)$, the neighbourhood axiom is trivially satisfied and the definition 2.2.4 is just stating that α is a clone homomorphism $O \to \mathrm{End}(A\langle 1\rangle)$. The definition thus coincides with the definition of a total O-algebra for a Set-clone O given in remark 1.3.16.

Note that the family $C(A\langle n\rangle, A\langle 1\rangle)$ doesn't admit a structure of a clone with $*_{nm}$ being defined by composition in C, in general. This is why we cannot simply say in definition 2.2.4 that α is a clone homomorphism. However, the family $E(n) = C(A\langle n\rangle, A\langle 1\rangle)$ does admit a structure of what could be called a *partial clone*. For each $n \in \mathbb{N}$ define $E(n)\langle m\rangle$ as $C(A\langle n\rangle, A\langle m\rangle) \rightarrowtail C(A\langle n\rangle, A\langle 1\rangle)^m$. Each $E(n)\langle - \rangle$ is an i-structure over $E(n)$. Define π_j^n as the projections $\mathrm{pr}_j : A\langle n\rangle \to A\langle 1\rangle$ and $*_{nm} : E(n) \times E(m)\langle n\rangle \to E(m)$ by (multi-)composition in C like it was done for an endomorphism clone in example 1.3.14(a). The $*_{nm}$ satisfy the associativity, projection and normalisation axiom in the definition of a clone when the axioms are suitably amended to fit the i-structures.

If we were to generalise clones and their homomorphisms to partial clones and homomorphisms of partial clones, respectively, then definition 2.2.4 would be equivalent to saying that $\alpha : O \to E$ is a homomorphism of partial clones in Set.

(b) Let C be an infinitary distributive category. The observation we made in remark 1.3.20 clearly extends to i-algebras and definition 2.2.4. Let $F : \mathrm{Set} \to C$ denote the product-preserving functor of taking copowers with the terminal object in C. The universal property of copowers yields a bijection of morphisms

$$O(n)1 \times A\langle n\rangle \to A\langle 1\rangle \quad \xleftrightarrow{1:1} \quad O(n) \to C(A\langle n\rangle, A\langle 1\rangle)$$

which restricts to a bijection of i-$F_*(O)$-algebras and i-O-algebras, and yields an isomorphism of $F_*(O)$ -IAlg(C) and O -IAlg(C) as categories over C. In particular, for $C = \mathrm{Set}$ this shows that definition 2.2.4 and definition 2.2.1 are equivalent.

(c) Let C be a category with finite products, A an i-structure in C and O_I the clone of the (algebraic) theory of equality in Set. The maps $\iota_n : O_I(n) \to C(A\langle n\rangle, A\langle 1\rangle)$ that map the variables x_j to pr_j for $n \geq 1$, together with the unique map $\iota_0 : \emptyset \to C(A\langle 0\rangle, A\langle 1\rangle)$ make A into an i-O_I-algebra. The neighbourhood and composition axioms follow from (3) in definition 2.1.1. The projection axiom is satisfied by construction. I-morphisms are easily seen to be i-O_I-homomorphisms. This construction thus yields an inverse to the forgetful functor O_I -IAlg$(C) \to$ I-Str(C) and shows in particular that both categories are isomorphic as categories over C.

Propositions 1.3.7 and 1.3.8 have straightforward extensions to i-algebras.

Proposition 2.2.6. *Let C be a category with finite products, and O a clone in C.*

(1) *A clone homomorphism $f : O \to O'$ induces a functor $f^* : O'$ -IAlg$(C) \to O$ -IAlg(C) of categories over C*

$$\begin{tikzcd} O' \text{-IAlg}(C) \arrow[rr, "f^*"] \arrow[dr, "U"'] & & O \text{-IAlg}(C) \arrow[dl, "U"] \\ & C & \end{tikzcd}$$

The map $f \mapsto f^*$ is functorial in $\mathrm{clone}(C)^{op}$, *i.e.* $(fg)^* = g^* \circ f^*$ *and* $1_O^* = I_{O\text{-IAlg}(C)}$.

(2) *Let C' be a category with finite products, and $F : C \to C'$ a finite-product preserving functor. If F preserves monomorphisms then F induces a functor $F_* : O$ -IAlg$(C) \to F_*(O)$ -IAlg(C') such that the subsequent diagram commutes*

$$\begin{tikzcd} O \text{-IAlg}(C) \arrow[r, "F_*"] \arrow[d, "U"'] & F_*(O) \text{-IAlg}(C') \arrow[d, "U'"] \\ C \arrow[r, "F"] & C' \end{tikzcd}$$

Proof. (1) Let (A, α) be an O'-algebra. Define $f^*\alpha$ to be the family of C-morphisms $\alpha_n \circ (f_n \times 1_{A\langle n\rangle})$, and define $f^*(A, \alpha)$ as $(A, f^*\alpha)$. $(A, f^*\alpha)$ is clearly an i-O-algebra. An i-O'-homomorphism $h : (A, \alpha) \to (A', \alpha')$ is easily seen an i-O-homomorphism $(A, f^*\alpha) \to (A', f^*\alpha')$. The construction is clearly functorial as asserted.

(2) By proposition 1.3.3 F induces a functor $F_* : \mathrm{clone}(C) \to \mathrm{clone}(C')$. In particular, the componentwise defined $F_*(O)$ is a clone in C'. Since F preserves products and monomorphisms it induces a functor by post-composition $F_* : \text{I-Str}(C) \to \text{I-Str}(C')$. If (A, α) is an i-O-algebra then $(F_*, F(\alpha))$ is clearly an i-$F_*(O)$-algebra. Moreover, if h is an i-homomorphism of i-O-algebras, then $F(h)$ is an i-homomorphism of the respective i-$F_*(O)$-algebras.

□

In remark 1.3.11 we have noted that clones in finite-product categories are precisely the models of the many-sorted algebraic theory of clones. The axioms are the equations given in remark 1.3.10 in their respective canonical context. (See also the clone axioms listed in remark 1.4.2.) Hence it is clear that in a finite-limit category C a collection of objects $O(n)$ for each $n \in \mathbb{N}$, morphisms $*_{nm} : O(n) \times O(m)^n \to O(m)$ for each $(n, m) \in \mathbb{N}^2$ and global elements $\pi^n_j : 1 \to O(n)$ for $n \geq 1$ and $1 \leq j \leq n$ is the data of a clone if and only if the equations-in-context given in remark 1.3.10 are satisfied in C in its internal language. We give a description of infinitesimal O-algebras in terms of the internal language.

Proposition 2.2.7. *Let C be a category with finite limits and O a clone in C.*

(*i*) *An i-structure A in C together with a family of morphisms $\bullet_n : O(n) \times A\langle n\rangle \to A\langle 1\rangle$, $n \in \mathbb{N}$ is an i-O-algebra if and only if C satisfies the following sequents*

(1) (*Neighbourhood*) *For each pair $(n, m) \in \mathbb{N}^2$, $n \geq 1$*

$$\top \vdash_{\vec{\sigma},x} A\langle n\rangle(\sigma_1 \bullet_m x, \ldots, \sigma_n \bullet_m x)$$

where x is of sort $A\langle m\rangle$ and each σ_k is of sort $O(m)$. Note that in the case $m = 0$ the sequent becomes $\top \vdash_{\vec{\sigma}} A\langle n\rangle(\bullet_0(\sigma_1), \ldots, \bullet_0(\sigma_n))$.

In addition, if the above sequent is valid in C then the formula-in-context

$$\vec{\sigma}, x, \vec{y}.(\vec{y} = (\sigma_1 \bullet_m x, \ldots, \sigma_n \bullet_m x)) \wedge A\langle n\rangle(y_1, \ldots, y_n), \qquad y_j : A\langle 1\rangle$$

is provably functional from $O(m)^n \times A\langle m\rangle$ to $A\langle n\rangle$ in the internal language of C, and hence corresponds to a C-morphism $i_{mn} : O(m)^n \times A\langle m\rangle \to A\langle n\rangle$ such that the *interpretation of the formula-in-context in C is the graph of*

$$O(m)^n \times A\langle m\rangle \xrightarrow{i_{mn}} A\langle n\rangle \rightarrowtail A\langle 1\rangle^n$$

(2) (*Associativity*) *For each* $(n, m) \in \mathbb{N}^2$

$$\top \vdash_{\sigma,\vec{t},x} \ \sigma \bullet_n (i_{mn}((t_1, \ldots, t_n), x)) = (\sigma *_{nm} (t_1, \ldots, t_n)) \bullet_m x$$

where x is of sort $A\langle m\rangle$, each t_k is of sort $O(m)$, σ of sort $O(n)$, and i_{nm} is the morphism from (1). *In the case $n = 0$ this becomes*

$$\top \vdash_{\sigma,x} \ \bullet_0(\sigma) = *_{0m}(\sigma) \bullet_m x$$

and in the case $m = 0$

$$\top \vdash_{\sigma,\vec{t}} \ \sigma \bullet_n (i_{0n}(t_1, \ldots, t_n)) = \bullet_0(\sigma *_{n0} (t_1, \ldots, t_n))$$

(3) (*Projection*) *For each $n \geq 1$ and $1 \leq j \leq n$*

$$\top \vdash_{x:A\langle n\rangle} \ \pi^n_j \bullet_n (x) = \mathrm{pr}_j(x)$$

(*ii*) *An i-morphism $h : A\langle 1\rangle \to A'\langle 1\rangle$ is an i-O-homomorphism $(A, \bullet) \to (A', \bullet')$ if and only if C satisfies*

$$\top \vdash_{\sigma,x} h(\sigma \bullet_n x) = \sigma \bullet'_n h^n(x)$$

for each $n \in \mathbb{N}$, where x is of sort $A\langle n\rangle$, σ is of sort $O(n)$, and h^n is the name of the morphism $h^n : A\langle n\rangle \to A'\langle n\rangle$

Proof. (i) (1) The interpretation of the atomic formula-in-context on the right-hand side of the sequent is the pullback

$$\begin{tikzcd}
\llbracket \vec{\sigma}, x.A\langle n\rangle(\sigma_1 \bullet_m x, \ldots, \sigma_n \bullet_m x)\rrbracket \arrow[r] \arrow[d, tail] & A\langle n\rangle \arrow[d, tail] \\
O(m)^n \times A\langle m\rangle \arrow[r, "(\bullet_m \circ (\mathrm{pr}_1 \times 1_{A\langle m\rangle}), \ldots, \bullet_m \circ (\mathrm{pr}_n \times 1_{A\langle m\rangle}))"] & A\langle 1\rangle^n
\end{tikzcd}$$

The sequent is satisfied in C if and only if the monomorphism on the left is an isomorphism. This is the case if and only if the morphism $\alpha_{mn} = (\bullet_m \circ (\mathrm{pr}_1 \times 1_{A\langle m\rangle}), \ldots, \bullet_m \circ (\mathrm{pr}_n \times 1_{A\langle m\rangle}))$

factors through $A\langle n\rangle$; but α_{mn} is the same morphism as

$$O(m)^n \times A\langle m\rangle \xrightarrow{1_{O(m)^n}\times\Delta_{A\langle m\rangle}} O(m)^n \times (A\langle m\rangle)^n \xrightarrow{\cong} (O(m)\times A\langle m\rangle)^n \xrightarrow{(\bullet_m)^n} A\langle 1\rangle^n$$

This shows that (1) is equivalent to the neighbourhood axiom in definition 2.2.1. We shall use $\vec{\sigma}, x, \vec{y}.\theta$ to denote the formula-in-context

$$\vec{\sigma}, x, \vec{y}.(\vec{y} = (\sigma_1 \bullet_m x, \ldots, \sigma_n \bullet_m x)) \wedge A\langle n\rangle(y_1, \ldots, y_n)$$

We show the assertion about the morphism i_{mn}. (This will also show θ provably functional.) The interpretation $[\![\vec{\sigma}, x, \vec{y}.(\vec{y} = (\sigma_1 \bullet_m x, \ldots, \sigma_n \bullet_m x))]\!]$ is the graph of α_{mn}

$$(1_{(O(m)^n\times A\langle m\rangle)}, \alpha_{mn}) : O(m)^n \times A\langle m\rangle \rightarrowtail O(m)^n \times A\langle m\rangle \times A\langle 1\rangle^n$$

The interpretation of θ is given by the pullback square

$$\begin{tikzcd}
{[\![\vec{\sigma}, x, \vec{y}.\theta]\!]} \arrow[rr, tail] \arrow[d, tail] & & O(m)^n \times A\langle m\rangle \arrow[d, tail, "(1_{(O(m)^n\times A\langle m\rangle)}, \alpha_{mn})"] \arrow[dll, dashed, "(1_{(O(m)^n\times A\langle m\rangle)}, i_{mn})"'] \\
O(m)^n \times A\langle m\rangle \times A\langle n\rangle \arrow[rr, tail, "1_{(O(m)^n\times A\langle m\rangle)}\times j_n"] & & O(m)^n \times A\langle m\rangle \times A\langle 1\rangle^n
\end{tikzcd}$$

where j_n denotes the monomorphism $A\langle n\rangle \rightarrowtail A\langle 1\rangle^n$. From the first part of the proof we know that α_{mn} factors through $A\langle n\rangle$. We define i_{nm} as the (left) morphism of this factorisation. The graph of i_{mn} yields the dashed diagonal arrow in the diagram above. Since the diagram is a pullback the top horizontal monomorphism splits (by a monomorphism) and is thus an isomorphism. The interpretation of $[\![\vec{\sigma}, x, \vec{y}.\theta]\!]$ is hence given by the graph of α_{nm}, which is also the graph of $j_n \circ i_{mn}$ as asserted.
For θ to be a functional relation from $O(m)^n \times A\langle m\rangle$ to $A\langle n\rangle$ the sequents

$$\theta \vdash_{\vec{\sigma},x,\vec{y}} A\langle n\rangle(y_1, \ldots, y_n) \qquad \text{and} \qquad \top \vdash_{\vec{\sigma},x} (\exists!\vec{y})\theta$$

have to be satisfied by C. The first one is satisfied by construction (apply the conjunction rule), the second is equivalent to saying that the composite

$$[\![\vec{\sigma}, x, \vec{y}.\theta]\!] \rightarrowtail O(m)^n \times A\langle m\rangle \times A\langle n\rangle \xrightarrow{\text{pr}} O(m)^n \times A\langle m\rangle$$

is an isomorphism, which follows from its interpretation as a graph.
The sequents in (2) and (3) are saying that the composites of the respective morphisms are equal (see definition 1.1.11 and [Joh02, lem. D1.3.1]). This is equivalent to saying that the respective associativity and projection diagrams in definition 2.2.1 commute, where the projection pr_j is once again understood as the respective restriction $\mathrm{pr}_j \circ j_n$ of the projection onto the jth factor.

(ii) This is similar to (2) and (3) in (i).

□

We would have liked to write $\sigma \bullet_n (t_1 \bullet_m x, \dots, t_n \bullet_m x)$ for the term $\sigma \bullet_n (i_{mn}((t_1, \dots, t_n), x))$ and $\pi^n_j \bullet_n (x_1, \dots, x_n) = x_j$ for the formula $\pi^n_j \bullet_n (x) = \mathrm{pr}_j(x)$. However, since $\bullet_m$ is a function symbol with type $O(m)A\langle m\rangle \to A\langle 1\rangle$ and not $O(m)A\langle 1\rangle^m \to A\langle 1\rangle$ we cannot form those terms formally. Such a notation can only be used informally in the framework of categorical first-order logic we use.

2.3 Syntactic infinitesimally affine spaces

In chapter 2.1.1 we have seen that every object in the syntactic category $C_{\mathbb{T}}$ of the theory $\mathbb{T}$ of (commutative, unital) K-algebras carries a nil-square i-structure, and that every morphism between objects corresponding to Horn formulae-in-context becomes an i-morphism. In lemma 1.3.22 we have shown that there is a clone $\mathcal{A}$ of affine combinations in $C_{\mathbb{T}}$. We now wish to show that every such nil-square i-structure admits a natural structure of an i-$\mathcal{A}$-algebra, and that every i-morphism becomes an i-$\mathcal{A}$-homomorphism. These results extend naturally to the cartesian formulae-in-context $\vec{x}.(\exists!\vec{y})\phi$ containing an existential quantifier for which we have shown that the isomorphism $\{\vec{x}, \vec{y}.\phi\} \to \{\vec{x}'.(\exists!\vec{y})\phi[\vec{x}'/\vec{x}]\}$ represented by $\vec{y}, \vec{x}, \vec{x}'.\phi \wedge (\vec{x} = \vec{x}')$ is an isomorphism of nil-square i-structures.

The i-$\mathcal{A}$-algebras in $C_{\mathbb{T}}$ and the i-$\mathcal{A}_R$-algebras in a finite-limit category C with a K-algebra object R are of key importance to applications of infinitesimal algebra to SDG. We call them **infinitesimally affine spaces** (or *i-affine spaces* for short) and the respective i-homomorphism **infinitesimally affine maps** (or *i-affine maps* for short). The category $\mathcal{A}_R$-IAlg(C) of i-affine spaces and i-affine maps will be denoted by $\mathrm{IAff}(C)$. If we want to stress the K-algebra object R, we will speak about *i-affine spaces over R*.

Theorem 2.3.1 (*Syntactic i-affine spaces*)**.** *Let $\mathbb{T}$ be the algebraic theory of* (*commutative, unital*) *K-algebras, $C_{\mathbb{T}}$ its syntactic category and $\mathring{C}_{\mathbb{T}}$ the full subcategory generated by the objects corresponding to Horn formulae-in-context. The forgetful functor $U : \mathrm{IAff}(C_{\mathbb{T}}) \to C_{\mathbb{T}}$*

has a section $Af : \mathring{C}_{\mathbb{T}} \to \mathrm{IAff}(C_{\mathbb{T}})$ *(i.e.* $U \circ Af = I_{\mathring{C}_{\mathbb{T}}}$*) that maps each object* $\{\vec{x}.\phi\}$ *to the i-affine space induced by the nil-square i-structure over* $\{\vec{x}.\phi\}$*.*

Proof. (1) Let A denote the unique sort of the signature Σ of K-algebras. We begin by considering the objects $A^n = \{\vec{x}.\top\}$. A^n is a total $\mathcal{A}$-algebra by purely abstract reasons. Firstly, since A^n is $\mathcal{L}(n)$ for the clone of linear combinations in $C_{\mathbb{T}}$, it is the 'free $\mathcal{L}$-algebra on n generators' (see chapter 1.3.3, in particular remark 1.3.13, and chapter 1.3.5 for $\mathcal{L}$). $\mathcal{A}$ is a subclone of $\mathcal{L}$. By proposition 1.3.7 the inclusion induces an $\mathcal{A}$-algebra structure on A^n. This $\mathcal{A}$-structure is given explicitly as follows.

Let α_m be the formula $1 = x_1 + \ldots + x_m$ and ∂_n the formula $\bigwedge_{1\le i,j\le n}(x_i - y_i)(x_j - y_j) = 0$. The morphism $\bullet_m : \mathcal{A}(m) \times A^n\langle m\rangle \to A^n$ is the $\mathbb{T}$-provable equivalence class of the formula-in-context

$$\vec{\lambda}, \vec{x}^{\,1}, \ldots, \vec{x}^{\,m}, \vec{y}.(\vec{y} = \lambda_1\vec{x}^{\,1} + \ldots + \lambda_m\vec{x}^{\,m}) \wedge \alpha_m[\vec{\lambda}/\vec{x}] \wedge \bigwedge_{1\le k,\ell\le m} \partial_n[\vec{x}^{\,k}, \vec{x}^{\,\ell}/\vec{x}, \vec{y}]$$

To show that the nil-square i-structure over A^n makes A^n into an i-affine space it is sufficient to show the neighbourhood axiom. From this it will follow that the total $\mathcal{A}$-algebra structure restricts to the nil-square i-structure, and thus show A^n an i-affine space as asserted.

As it is common practice in mathematics we will speak of $(\vec{y} = \lambda_1\vec{x}^{\,1} + \ldots + \lambda_m\vec{x}^{\,m}) \wedge \alpha_m[\vec{\lambda}/\vec{x}]$ simply as an *affine combination of points*, write $\vec{x} \sim_1 \vec{y}$ for ∂_m and say that $\vec{x}$ and $\vec{y}$ are (1-)*neighbours*. By lemma 2.1.14(i)(1) (and the fact that nil-square i-structures are generated by the first neighbourhood of the diagonal) the neighbourhood axiom amounts to show that any two affine combinations $\lambda_1\vec{x}^{\,1} + \ldots + \lambda_m\vec{x}^{\,m}$ and $\mu_1\vec{x}^{\,1} + \ldots + \mu_m\vec{x}^{\,m}$ are neighbours, if $\vec{x}^{\,k} \sim_1 \vec{x}^{\,\ell}$ for $1 \le k, \ell \le m$. More formally, and in terms of sequents, we have to show the sequent

$$\bigwedge_{1\le k,\ell\le m} \partial_n[\vec{x}^{\,k}, \vec{x}^{\,\ell}/\vec{x}, \vec{y}] \;\wedge\; \alpha_m[\vec{\lambda}/\vec{x}] \;\wedge\; \alpha_m[\vec{\mu}/\vec{x}] \quad \vdash_{\vec{\lambda},\vec{\mu},\vec{x}^{1},\ldots,\vec{x}^{m}} \quad \partial_n\Big[\sum_{j=1}^m \lambda_j\vec{x}^{\,j}, \sum_{j=1}^m \mu_j\vec{x}^{\,j}/\vec{x}, \vec{y}\Big]$$

$\mathbb{T}$-provable. We proceed in two steps. Let $\vec{x}^{\,k} \sim_1 \vec{x}^{\,\ell}$ for $1 \le k, \ell \le m$. Firstly, we show that if $\vec{y} \sim_1 \vec{x}^{\,k}$ for $1 \le k \le m$, then $\vec{y} \sim_1 \sum_{k=1}^m \lambda_k\vec{x}^{\,k}$. Using $\sum_{k=1}^m \lambda_k = 1$ we find

$$\begin{aligned}\Big(\sum_{k=1}^m \lambda_k x_i^k - y_i\Big)\Big(\sum_{k=1}^m \lambda_k x_j^k - y_j\Big) &= \Big(\sum_{k=1}^m \lambda_k(x_i^k - y_i)\Big)\Big(\sum_{k=1}^m \lambda_k(x_j^k - y_j)\Big)\\ &= \sum_{k,\ell=1}^m \lambda_k\lambda_\ell(x_i^k - y_i)(x_j^\ell - y_j)\end{aligned}$$

From $(x_i^k - x_i^\ell)(x_j^k - x_j^\ell) = 0$ we obtain

$$\begin{aligned} 0 &= ((x_i^k - y_i) + (y_i - x_i^\ell))((x_j^k - y_j) + (y_j - x_j^\ell)) \\ &= (x_i^k - y_i)(x_j^k - y_j) + (x_i^k - y_i)(y_j - x_j^\ell) + (y_i - x_i^\ell)(x_j^k - y_j) + (y_i - x_i^\ell)(y_j - x_j^\ell) \\ &= (x_i^k - y_i)(y_j - x_j^\ell) + (y_i - x_i^\ell)(x_j^k - y_j) \end{aligned}$$

(We have used $\vec{y} \sim_1 \vec{x}^{\,k}$ in the last step.) This yields

$$(x_i^k - y_i)(x_j^\ell - y_j) = -(x_i^\ell - y_i)(x_j^k - y_j)$$

and hence, due to $\lambda_k\lambda_\ell = \lambda_\ell\lambda_k$,

$$\sum_{k,\ell=1}^{m} \lambda_k\lambda_\ell(x_i^k - y_i)(x_j^\ell - y_j) = 0$$

We apply this result to $\vec{y} = \vec{x}^\ell$ for each $1 \le \ell \le m$ first and obtain $\vec{x}^{\,\ell} \sim_1 \sum_{k=1}^m \lambda_k\vec{x}^{\,k}$. Next we apply it to $\vec{y} = \sum_{k=1}^m \lambda_k\vec{x}^{\,k}$ and conclude

$$\sum_{\ell=1}^{m} \mu_\ell\vec{x}^{\,\ell} \sim_1 \sum_{k=1}^{m} \lambda_k\vec{x}^{\,k}$$

as desired.

(2) In this step we wish to show that for each a Horn formula ϕ, $\{\vec{x}.\phi\}$ together with its nil-square i-structure can be made into an i-affine space. In fact, we want to show $\{\vec{x}.\phi\}$ an i-affine subspace of A^n (where n is the length of the context $\vec{x}$). The morphisms $\bullet_m : \mathcal{A}(m) \times \{\vec{x}.\phi\}\langle m\rangle \to \{\vec{x}.\phi\}$ shall thus be represented by the formula-in-context

$$\vec{\lambda}, \vec{x}^1, \ldots, \vec{x}^{\,m}, \vec{y}.(\vec{y} = \sum_{k=1}^{m} \lambda_k\vec{x}^{\,k}) \wedge \alpha_m[\vec{\lambda}/\vec{x}] \wedge \bigwedge_{1\le k,\ell\le m} \partial_n[\vec{x}^{\,k}, \vec{x}^{\,\ell}/\vec{x}, \vec{y}] \wedge \bigwedge_{1\le k\le m} \phi[\vec{x}^{\,k}/\vec{x}]$$

We only need to show that the codomain of this morphism is indeed $\{\vec{x}.\phi\}$, i.e. that the sequent

$$(\vec{y} = \sum_{k=1}^{m} \lambda_k\vec{x}^{\,k}) \wedge \alpha_m[\vec{\lambda}/\vec{x}] \wedge \bigwedge_{1\le k,\ell\le m} \partial_n[\vec{x}^{\,k}, \vec{x}^{\,\ell}/\vec{x}, \vec{y}] \wedge \bigwedge_{1\le k\le m} \phi[\vec{x}^{\,k}/\vec{x}] \vdash_{\vec{\lambda}, \vec{x}^1, \ldots, \vec{x}^{\,m}, \vec{y}} \phi[\vec{y}/\vec{x}] \quad (2.4)$$

is provable in $\mathbb{T}$. In other words, if for a family of mutual neighbours $\vec{x}^k$ each one satisfies ϕ, then their affine combination also satisfies ϕ. We will proceed by induction over the structure of ϕ.

As we mentioned before every term-in-context $\vec{x}.t$ over the signature Σ of K-algebras is $\mathbb{T}$-provably equal to a polynomial in $K[\vec{x}]$; that is, there is a polynomial $P \in K[\vec{x}]$ and the sequent $\top \vdash_{\vec{x}} P = t$ is provable in $\mathbb{T}$. An equation $s = t$ is thus $\mathbb{T}$-provably equivalent to $P = 0$, where the term P is a polynomial in the variables of the canonical context that is $\mathbb{T}$-provably equal to the term $s - t$. Every Horn formula-in-context over Σ is hence $\mathbb{T}$-provably equivalent to one of the form $\vec{x}.\psi$, where ψ is a conjuction of equations of the form $P = 0$ for polynomials P in $K[\vec{x}]$. The sequent (2.4) is provable in $\mathbb{T}$, if any sequent that can be obtained from (2.4) by replacing $\vec{x}.\phi$ with a $\mathbb{T}$-provably equivalent formula-in-context, is provable in $\mathbb{T}$.

We show that if ϕ is $P = 0$ then the sequent (2.4) is $\mathbb{T}$-provable. This follows from the following sequent stating that polynomials preserve i-affine combinations

$$\alpha_m[\vec{\lambda}/\vec{x}] \wedge \bigwedge_{1 \leq k,\ell \leq m} \partial_n[\vec{x}_k, \vec{x}_\ell/\vec{x}, \vec{y}] \vdash_{\vec{\lambda},\vec{x}_1,\ldots,\vec{x}_m} P\Big(\sum_{k=1}^{m} \lambda_k \vec{x}_k\Big) = \sum_{k=1}^{m} \lambda_k P(\vec{x}_k)$$

The polynomial version of Hadamard's lemma (lemma 2.1.4) implies that this sequent is provable in $\mathbb{T}$. Indeed, by Hadamard's lemma there are polynomials $L_i \in K[\vec{x}]$ and $Q_{ij} \in K[\vec{x}, \vec{y}]$ such that

$$P(\vec{x}_\ell) - P(\vec{x}) = \sum_{i=1}^{n} (x_{\ell,i} - x_i) L_i(\vec{x}) + \sum_{i,j=1}^{n} (x_{\ell,i} - x_i)(x_{\ell,j} - x_j) Q_{ij}(\vec{x}, \vec{x}_\ell)$$

If $\vec{x}_\ell \sim_1 \vec{x}$ then the equation simplifies to

$$P(\vec{x}_\ell) - P(\vec{x}) = \sum_{i=1}^{n} (x_{\ell,i} - x_i) L_i(\vec{x})$$

In (1) we have shown that $\vec{x}_\ell \sim_1 \sum_{k=1}^m \lambda_k \vec{x}_k$. Therefore, by using this and $\sum_{\ell=1}^m \lambda_\ell = 1$ we compute

$$\begin{aligned}
\sum_{\ell=1}^m \lambda_\ell P(\vec{x}_\ell) - P\big(\sum_{k=1}^m \lambda_k \vec{x}_k\big) &= \sum_{\ell=1}^m \lambda_\ell \big(P(\vec{x}_\ell) - P\big(\sum_{k=1}^m \lambda_k \vec{x}_k\big)\big) \\
&= \sum_{\ell=1}^m \lambda_\ell \sum_{i=1}^n \big(x_{\ell,i} - \sum_{k=1}^m \lambda_k x_{k,i}\big) L_i\big(\sum_{k=1}^m \lambda_k \vec{x}_k\big) \\
&= \sum_{i=1}^n \big(\sum_{\ell=1}^m \lambda_\ell x_{\ell,i} - \sum_{\ell=1}^m \lambda_\ell \sum_{k=1}^m \lambda_k x_{k,i}\big) L_i\big(\sum_{k=1}^m \lambda_k \vec{x}_k\big) \\
&= \sum_{i=1}^n \big(\sum_{\ell=1}^m \lambda_\ell x_{\ell,i} - \sum_{k=1}^m \lambda_k x_{k,i}\big) L_i\big(\sum_{k=1}^m \lambda_k \vec{x}_k\big) \\
&= 0
\end{aligned}$$

Now if ϕ is any conjunction of equations of the form $P = 0$ for a polynomial P then, since (2.4) is $\mathbb{T}$-provable for each such equation, their conjunction ϕ is also $\mathbb{T}$-provable.

(3) Next we wish to show that every morphism in $\mathring{C}_\mathbb{T}$ is i-affine. We shall do this by working in the equivalent category $K\text{-Alg}_{fp}^{op}$ of finitely presented K-algebras. The notation is the same as in chapter 2.1.1, when we discussed the nil-square i-structures. We need to show that for any K-algebra homomorphism $f : K[\vec{X}]/I \to K[\vec{Y}]/J$ the subsequent diagram commutes

$$\begin{array}{ccc}
K[\vec{X}]/I & \xrightarrow{\alpha_m} & K[\vec{\lambda}, \vec{X}_1, \ldots, \vec{X}_m]/((\sum_{j=1}^m \lambda_j - 1) \cup mI \cup I_{\partial_{n,m}}) \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle 1_{K[\vec{\lambda}]/(\sum_{j=1}^m \lambda_j - 1)} \otimes_K f^{\otimes m}} \\
K[\vec{Y}]/J & \xrightarrow{\beta_m} & K[\vec{\lambda}, \vec{Y}_1, \ldots, \vec{Y}_m]/((\sum_{j=1}^m \lambda_j - 1) \cup mJ \cup I_{\partial_{k,m}})
\end{array} \tag{2.5}$$

The K-algebra homomorphisms α_m and β_m are defined on generators as follows

$$\alpha_m([X_i]) = \sum_{j=1} [\lambda_j X_{j,i}] \qquad \text{and} \qquad \beta_m([Y_i]) = \sum_{j=1} [\lambda_j Y_{j,i}]$$

They represent the translated i-affine structures in $K\text{-Alg}_{fp}^{op}$. Note that all algebra homomorphisms are well-defined, for we have shown f an i-morphism already. Since every f has a lift $\bar{f} : K[\vec{X}] \to K[\vec{Y}]$ along the quotient maps $K[\vec{X}] \twoheadrightarrow K[\vec{X}]/I$ and $K[\vec{Y}] \twoheadrightarrow K[\vec{X}]/I$, it is sufficient to consider the case that I and J are the zero ideals. In this case (2.5) commutes if and only if it commutes for the generators

$$\big[P_i\big(\sum_{j=1}^{m}\lambda_j\vec{Y}_j\big)\big] = \sum_{j=1}^{m}[\lambda_j P_i(\vec{Y}_j)], \qquad 1 \leq i \leq n$$

where $P_i(\vec{Y}) = f(X_i)$. This amounts to show

$$P_i\big(\sum_{j=1}^{m}\lambda_j\vec{Y}_j\big) - \sum_{j=1}^{m}\lambda_j P_i(\vec{Y}_j) \in \big(\big(\sum_{j=1}^{m}\lambda_j - 1\big) \cup I_{\partial_{n,m}}\big)$$

which is the same calculation we have done in (2) to show that every polynomial preserves i-affine combinations in $C_{\mathbb{T}}$; it follows from Hadamard's lemma for polynomials.

With this we may finally conclude that every object $\{\vec{x}.\phi\}$ in $\mathring{C}_{\mathbb{T}}$ has a canonical structure of an i-affine space on its nil-square i-structure and that every morphism becomes an i-affine map. This yields the desired functor $Af : \mathring{C}_{\mathbb{T}} \longrightarrow \mathrm{IAff}(C_{\mathbb{T}})$, which is clearly a section of U, as asserted. □

The result of theorem 2.3.1 extends naturally to cartesian formulae-in-context $\vec{x}.(\exists!\vec{y})\phi$ containing an existential quantifier for which we have shown that the isomorphism $\{\vec{x}, \vec{y}.\phi\} \longrightarrow \{\vec{x}'.(\exists!\vec{y})\phi[\vec{x}'/\vec{x}]\}$ represented by $\vec{y}, \vec{x}, \vec{x}'.\phi \wedge (\vec{x} = \vec{x}')$ is an isomorphism of nil-square i-structures.

Theorem 2.3.2. *Let $\vec{x}.(\exists!\vec{y})\phi$ be a cartesian formula-in-context and ϕ Horn. If the isomorphism*

$$f : \{\vec{x}, \vec{y}.\phi\} \longrightarrow \{\vec{x}'.(\exists!\vec{y})\phi[\vec{x}'/\vec{x}]\}$$

represented by $\vec{y}, \vec{x}, \vec{x}'.\phi \wedge (\vec{x} = \vec{x}')$ is an isomorphism of nil-square i-structures, then it is also an isomorphism of the i-affine spaces, where $\{\vec{x}.(\exists!\vec{y})\phi\} \rightarrowtail A^n$ carries the i-affine structure over its nil-square i-structure making it an i-affine subspace of A^n.

Proof. Due to the representation of the isomorphism f by the formula ϕ proving $\{\vec{x}.(\exists!\vec{y})\phi\}$ an i-affine subspace of A^n is equivalent to showing that f and its inverse are i-affine maps. We show the former, but present the argument in a less formal way.

Without loss of generality we may assume that ϕ is of the form $(P_1(\vec{x}, \vec{y}) = 0) \wedge \ldots \wedge (P_L(\vec{x}, \vec{y}) = 0)$. Suppose that there are i-neighbours $\vec{x}_1, \ldots, \vec{x}_m$, i.e. $\vec{x}_k \sim_1 \vec{x}_\ell$, $1 \leq k, \ell \leq m$, with the corresponding $\vec{y}_1, \ldots, \vec{y}_m$ such that $\phi[\vec{x}_k, \vec{y}_k/\vec{x}, \vec{y}]$ is satisfied. By assumtion f is an isomorphism of i-structures, so $(\vec{x}_k, \vec{y}_k) \sim_1 (\vec{x}_\ell, \vec{y}_\ell)$, $1 \leq k, \ell \leq m$. Since polynomials preserve i-affine combinations (see step (2) in the proof of 2.3.1), we have

$$P_j\big(\sum_{k=1}^{m} \lambda_k \vec{x}_k, \sum_{k=1}^{m} \lambda_k \vec{y}_k\big) = \sum_{k=1}^{m} \lambda_k P_j(\vec{x}_k, \vec{y}_k), \qquad 1 \leq j \leq L$$

This shows that $\sum_{k=1}^{m} \lambda_k \vec{x}_k$ and $\sum_{k=1}^{m} \lambda_k \vec{y}_k$ satisfy the polynomial equations defined by ϕ, and hence that $\{\vec{x}.(\exists! \vec{y})\phi\}$ is an i-affine subspace of A^n as asserted. □

Having the result for the syntactic category $C_{\mathbb{T}}$ gives us a supply of i-affine spaces and i-affine maps in every finite-limit category C with a K-algebra object R.

Corollary 2.3.3. *Let C be a category with finite limits, and R a K-algebra object in C. Let $F_R : C_{\mathbb{T}} \to C$ be the functor that maps the universal K-algebra $M_{\mathbb{T}}$ to R. We have that the following diagram commutes*

$$\begin{tikzcd} \mathrm{IAff}(C_{\mathbb{T}}) \arrow[r, "(F_R)_*"] & \mathrm{IAff}(C) \arrow[d, "U"] \\ \mathring{C}_{\mathbb{T}} \arrow[u, hook, "Af"] \arrow[r, "F_R"] & C \end{tikzcd}$$

where $(F_R)_*$ is the functor induced by F_R as in proposition 2.2.6(2).

In geometric terms, the spaces in C constructed out of R that are in the image of F_R are the *zero-loci* of finite families of polynomials and the morphism are the ‘regular’ morphisms that can be represented by finite families of polynomials. The corollary is thus stating that such loci in C are infinitesimally affine and the regular morphisms are i-affine maps.

Remark 2.3.4 (*Finitely generated K-algebras*)**.** Theorem 2.3.1 has a straightforward generalisation to the category $K\text{-Alg}_{fg}^{op}$. Indeed, step (1) and (2) of the proof of theorem 2.3.1 extend to $K\text{-Alg}_{fg}^{op}$, if we consider it as the syntactic category of K-algebras in infinitary cartesian logic. In that case the polynomial normal form of a formula can still be derived and will have infinite conjunctions of equations of the form $P = 0$ indexed by arbitrary sets, not just finite ones. However, as we have pointed out in the proof, if the sequent (2.4) is valid for any family of equations of the form $P = 0$, it is also valid for the conjunction over that family. Also, step (3) did not depend on the ideals being finitely generated, hence extends to finitely generated K-algebras.

This shows that finitely generated *affine schemes* are i-affine spaces in a natural way, and morphisms of affine schemes are i-affine maps. For a category C with finite limits, equalizers of small families of morphisms and a K-algebra object R, this extends the class of i-affine spaces and i-affine maps to zero-loci of arbitrary families of polynomials and the 'regular' maps admitting a presentation by a (finite) family of polynomial maps.

The argument can be extended to the opposite category of all K-algebras (in Set).

Theorem 2.3.5. *There is a functor* $Af : K\text{-Alg}^{op} \to \mathrm{IAff}(K\text{-Alg}^{op})$ *extending* $Af : K\text{-Alg}^{op}_{fp} \to \mathrm{IAff}(K\text{-Alg}^{op}_{fp})$*, i.e. the subsequent diagram commutes for the fully faithful functor* $\iota : K\text{-Alg}^{op}_{fp} \to K\text{-Alg}^{op}$

$$\begin{array}{ccc} K\text{-Alg}^{op} & \xrightarrow{Af} & \mathrm{IAff}(K\text{-Alg}^{op}) \\ \uparrow{\scriptstyle \iota} & & {\scriptstyle \iota_*}\uparrow \\ K\text{-Alg}^{op}_{fp} & \xrightarrow{Af} & \mathrm{IAff}(K\text{-Alg}^{op}_{fp}) \end{array}$$

and we have a natural isomorphism $U \circ Af \cong I_{K\text{-Alg}^{op}}$*, where* $U : \mathrm{IAff}(K\text{-Alg}^{op}) \to K\text{-Alg}^{op}$ *denotes the forgetful functor and* $K\text{-Alg}$ *stands for the category* $K\text{-Alg}(\mathrm{Set})$*.* (*Since we work with finitely presented K-algebras instead of finitely presentable ones, the functor* ι *is not injective on objects and hence not an embedding.*)

Proof. To begin with, recall that every K-algebra is a filtered colimit of finitely generated K-algebras. This is the conceptual reason for why the assertion of the theorem is true. It is contained implicitly in the argument we shall give here.

Let A be a K-algebra. We fix a presentation of A, i.e. a set of generators G and an ideal I such that $K[(X)_{g\in G}]/I \cong A$. The free K-algebra $K[(X)_{g\in G}]$ is given by the set of polynomials in finitely many variables in X with the usual addition and multiplication of polynomials. The definition of the nil-square i-structure has a straightforward generalisation from the presentation of a finitely generated algebra to $K[(X)_{g\in G}]/I$:

$$K[(X_1)_{g\in G}, \ldots, (X_m)_{g\in G}]/mI \twoheadrightarrow K[(X_1)_{g\in G}, \ldots, (X_m)_{g\in G}]/(mI \cup I_{\partial_{X,m}})$$

where mI is the ideal generated by the disjoint union of m copies of $I[X_i/X]$ (the ideal I where the variables in X_g are substituted by $X_{g,i}$ for all $g \in G$), and $I_{\partial_{G,m}}$ is the ideal generated by the set

$$\{(X_{g,i} - X_{g,j})(X_{g',i} - X_{g',j}) \mid g, g' \in G,\ 1 \le i, j \le m\}$$

Any K-algebra homomorphism $f : K[(X)_{g\in G}]/I \to K[(Y)_{h\in H}]/J$ has a lift $\bar{f} : K[(X)_{g\in G}] \to K[(Y)_{h\in H}]$ along the quotient maps. Furthermore, since we are dealing with polynomials in finitely many variables we can use Hadamard's lemma for polynomials to prove the factorisation of $f^{\otimes m}$

$$\begin{tikzcd}
K[(X_1)_{g\in G},\dots,(X_m)_{g\in G}]/mI \arrow[r, two heads] \arrow[d, "f^{\otimes m}"'] & K[(X_1)_{g\in G},\dots,(X_m)_{g\in G}]/(mI\cup I_{\partial_{G,m}}) \arrow[d, dashed] \\
K[(Y_1)_{h\in H},\dots,(Y_m)_{h\in H}]/mJ \arrow[r, two heads] & K[(Y_1)_{h\in H},\dots,(Y_m)_{h\in H}]/(mJ\cup I_{\partial_{H,m}})
\end{tikzcd}$$

in the same way as in the case of finitely presented K-algebras. This implies that the collection of subobjects in $K\text{-Alg}^{op}$ we have defined above indeed constitutes an i-structure over $K[(X)_{g\in G}]/I$ and that every f is an i-morphism in $K\text{-Alg}^{op}$. As regards the i-affine structure, we proceed in the same way. The K-algebra homomorphisms

$$\alpha_m : K[(X)_{g\in G}]/I \to K[\vec{\lambda},(X_1)_{g\in G},\dots(X_m)_{g\in G}]/((\textstyle\sum_{j=1}^m \lambda_j - 1)\cup mI\cup I_{\partial_{G,m}})$$

given by $[X_g]\mapsto[\sum_{k=1}^m \lambda_k X_{k,g}]$ for each $g\in G$, make the nil-square i-structure on $K[(X)_{g\in G}]/I$ into an i-affine space in $K\text{-Alg}^{op}$. The axioms can be checked on the generators. In the case $I=0$ the associativity and projection axiom are clear, if the neighbourhood axiom holds. The latter follows from the same calculation as done in (1) of the proof of theorem 2.3.1 (if we repeat them modulo $I_{\partial_{G,m}}$). For an arbitrary ideal I we need to show that the α_m are well-defined; i.e. that for $P(X_{g_1},\dots,X_{g_n})$ in I the polynomial $P(\sum_{k=1}^m \lambda_k X_{k,g_1},\dots,\sum_{k=1}^m \lambda_k X_{k,g_1})$ lies in the ideal $((\sum_{j=1}^m \lambda_j - 1)\cup mI\cup I_{\partial_{G,m}})$. But this follows from the calculation we did in (2) of the proof of theorem 2.3.1 showing that polynomials preserve i-affine combinations (again by working modulo the respective ideals). Finally, the fact that every K-algebra homomorphism becomes an i-affine map follows as in (3) of the proof of theorem 2.3.1.

Two presentations are mapped to the same K-algebra if and only if they have the same number of generators and the equations generate the same ideal. From this (and the definition of the i-affine structure in $K\text{-Alg}^{op}$) it is clear that the functor $Af : K\text{-Alg}^{op}\to \mathrm{IAff}(K\text{-Alg}^{op})$ we constructed is an extension of the Af on finitely presented K-algebras along ι, i.e. $\iota_*\circ Af = Af\circ\iota$. □

Remark 2.3.6. In terms of logic we could consider $K\text{-Alg}^{op}$ (up to equivalence of categories) as the syntactic category of the theory of K-algebras in a ‘truly’ infinitary cartesian logic, which also allows infinite contexts and infinite applications of the existential quantifier ‘∃!’ (see, for example, [AR94, chap. 5.B]).

In fact, we can try to repeat the construction of i-affine spaces of theorem 2.3.1 in the syntactic categories of K-algebras in the fragments of regular, coherent and geometric logic. As we have remarked before, the fragment of the logic of a formula-in-context does not matter for the definition of the nil-square i-structure on $\{\vec{x}.\phi\}$; however, which fragment the formula belongs to becomes important for which morphisms lift to i-morphisms and for the i-affine structure.

To prove the sequent (2.4) for disjunctions $\bigvee_{i\in I}\phi_i$ of Horn formulae ϕ in coherent or geometric logic, we have to apply the distributivity law. The resulting disjunction will have formulae containing pure conjunctions $\bigwedge_{1\leq k\leq m}\phi_i[\vec{x}^{\,k}/\vec{x}]$ and mixed ones $\bigwedge_{1\leq k\leq m}\phi_{i_k}[\vec{x}^{\,k}/\vec{x}]$. If the sequent

$$\partial_n \wedge \phi_i \ \vdash_{\vec{x},\vec{y}} \ \phi_i[\vec{y}/\vec{x}]$$

is $\mathbb{T}$-provable for each $i\in I$, then we can reduce the mixed conjunctions to pure ones, and hence obtain the sequent (2.4) from the disjunction rule and the fact that it holds for each ϕ_i.

We have to be careful with the initial object $\{\bot\}$ in $\mathcal{C}^{coh}_{\mathbb{T}}$ and $\mathcal{C}^{geom}_{\mathbb{T}}$, which cannot have an i-affine structure as we defined it. The reason is purely structural, rather than conceptual: as a strict initial object there is no morphism from $\mathcal{A}(0) = \{0 = 1\}$ to $\{\bot\}$, as long as both formulae are not provably equivalent; which is the case if and only if $\mathbb{T}$ is a theory of non-trivial K-algebras. (See also remark 1.3.25.) However, there are several ways to amend this. Firstly, we can consider the (coherent) theory of *non-trivial* K-algebras $\mathbb{T}\cup\{(0=1)\vdash\bot\}$ and its syntactic category instead. Secondly, we could understand i-affine spaces as models of the cartesian *theory of i-affine spaces* $\mathrm{i}\mathbb{A}$ introduced in section 2.5.1, and, finally, we could simply redefine $\mathcal{A}(0) = \{\bot\}$ in $\mathcal{C}^{coh}_{\mathbb{T}}$ and $\mathcal{C}^{geom}_{\mathbb{T}}$. In any case, since the nil-square-i-structure is total on $\{\bot\}$, it will become a total affine space for any of the three approaches. (See also proposition 1.3.9.)

In contrast with the cartesian existential quantifier it seems that we need to make further assumptions about ϕ as well to be able to show that $\{\vec{x}.(\exists\vec{y})\phi\} \rightarrowtail A^n$ is an i-affine subspace for a Horn formula ϕ in the syntactic category $\mathcal{C}^{reg}_{\mathbb{T}}$. For example, if for any family of mutual i-neighbours $\vec{x}_1,\ldots,\vec{x}_m$ we can find a family of points $\vec{y}_1,\ldots,\vec{y}_m$, such that $\phi(\vec{x}_\ell,\vec{y}_\ell)$ holds for each $1\leq\ell\leq m$, and all the $(\vec{x}_\ell,\vec{y}_\ell)$ together form a family of mutual infinitesimal neighbours, then we can prove the sequent (2.4) for $(\exists\vec{y})\phi$. More formally, we assume that for each $m\geq 1$

$$\bigwedge_{1\leq k,\ell\leq m}\partial_n[\vec{x}_k,\vec{x}_\ell/\vec{x},\vec{y}]\wedge\bigwedge_{1\leq k\leq m}\phi[\vec{x}_k,\vec{y}_k/\vec{x},\vec{y}]\ \vdash_{\vec{x}_1,\ldots,\vec{x}_m,\vec{y}_1,\ldots,\vec{y}_m}\ \bigwedge_{1\leq k,\ell\leq m}\partial_{n+n'}[(\vec{x}_k,\vec{y}_k),(\vec{x}_\ell,\vec{y}_\ell)/\vec{x},\vec{y}]$$

is $\mathbb{T}$-provable, where n' is the length of the contexts $\vec{y}_k$.

We can then consider the full subcategory $\mathring{\mathcal{C}}^{geom}_{\mathbb{T}}$ of $\mathcal{C}^{geom}_{\mathbb{T}}$ of objects corresponding to formulae-in-context, which become i-affine spaces over the respective nil-square i-structure, and study which morphisms become i-affine maps.

In geometric terms the last theorem states that affine schemes are i-affine spaces and their morphisms are i-affine maps. We also know that the nil-square i-structure is generated by the first neighbourhood of the diagonal, a notion which originated in algebraic geometry.

That one can form affine combinations of a finite set of generalised points (of the same type), which are mutual 1-neighbours, as well as that morphisms of affine schemes preserve the neighbourhood relation and such affine combinations, this has been shown in [Koc15] already. Kock's result can be obtained from ours by applying the hom-functor $\hom(-, C) : K\text{-Alg} \to \mathrm{Set}$, or equivalently, $\hom(C, -) : K\text{-Alg}^{op} \to \mathrm{Set}$ to the i-affine space A, the nil-square i-structure over A, and the clone of affine combinations $\mathcal{A}$ in $K\text{-Alg}^{op}$. Since $\hom(C, -)$ preserves limits we obtain the Set-clone of affine combinations over the K-algebra C and an i-affine space $\hom(A, C)$ over that K-algebra. Kock also shows the representability of 'being n mutual neighbouring points', which is isomorphic to our construction of the nil-square i-structure. The ideals he uses, however, do not rely on the choice of a presentation like ours. This is because Kock's approach makes use of the K-algebra structure directly, whereas we work with presentations of K-algebras due to the syntactic approach. His representation of 'forming affine combinations' differs from ours slightly, and hence does not correspond to the syntactically defined clone of affine combinations. Theorem 2.3.5 and our theory put Kock's result into the proper perspective by supplying the adequate definitions and structures.

More importantly, our approach via the syntactic category of K-algebras has further ramifications. One of them is the extension of theorem 2.1.5 to (certain subcategories of) syntactic categories of the algebraic theory of K-algebras in different fragments of first-order logic. Our proof based on Hadamard's lemma for polynomials will generalise straightforwardly to the case of C^∞-rings. In particular, we will see that manifolds (when embedded into $C^\infty\text{-Rng}_{fp}^{op}$) are i-affine spaces and smooth maps preserve that structure. This and other ramifications of this approach will be discussed in chapter 3.

2.4 Infinitesimalisation of algebraic theories

In this section we study the notion of an IMAT using presentations of algebraic theories in the syntactic approach. An infinitesimal model of an algebraic theory $\mathbb{T}$ will have all the operations of $\mathbb{T}$ become partial operations, which domain of definition are given by the i-structure. To describe this syntactically we will construct a cartesian theory $\mathrm{I}[\mathbb{T}]$ out of $\mathbb{T}$, its *infinitesimalisation*.

There are two equivalent ways of defining infinitesimalisation of an algebraic theory $\mathbb{T}$ in cartesian logic: the first is many-sorted and focuses on the function symbols and hence the operations of $\mathbb{T}$, the second is one-sorted and focuses on the i-structure and hence the relations. We define infinitesimalisation using the second approach, for it is more suitable for applications in practice. The first approach will be defined in remark 2.4.4 later.

Unfortunately, the language of first-order logic that we have introduced does not support partially defined operations and terms naturally. We will need to represent the partially defined operations by their graphs and hence translate terms into formulae, which will make the definition of infinitesimalisation rather cumbersome.

Definition 2.4.1 (*Infinitesimalisation*)**.** Let $\mathbb{T}$ be an algebraic theory over the signature Σ (with sort A, say). The **infinitesimalisation** $\mathrm{I}[\mathbb{T}]$ of $\mathbb{T}$ (or *i-sation* for short) is a cartesian theory over a signature $\mathrm{I}[\Sigma]$ defined as follows:

- $\mathrm{I}[\Sigma]$-Sort has one sort A.
- There are no function symbols.
- For each $n \in \mathbb{N}$, $\mathrm{I}[\Sigma]$ has an n-ary relation symbol

$$A\langle n\rangle \rightarrowtail A \cdots A$$

- For each n-ary function symbol f in Σ-Fun, there is an $(n+1)$-ary relation symbol

$$G_f \rightarrowtail A \cdots A$$

For each Σ-term-in-context $\vec{x}.t$ there is a formula-in-context $\vec{x}, y_t.\phi_t$ over $\mathrm{I}[\Sigma]$ (with $y_t : A$) that is defined by recursion over the structure of t:

- If t is a variable x_j, then ϕ_t is

$$(y_t = x_j) \wedge A\langle n\rangle(x_1, \ldots, x_n)$$

- If t is $f(t_1, \ldots, t_m)$ then ϕ_t is

$$(\exists! y_{t_1}) \cdots (\exists! y_{t_m}) G_f(y_{t_1}, \ldots, y_{t_n}, y_t) \wedge \bigwedge_{1 \le j \le m} \phi_{t_j}$$

(Note that the use of the existential quantifiers is sound relative to the axiom (2) below.)

The axioms of $\mathrm{I}[\mathbb{T}]$ are as follows:

(1) (**I-structure**) $\top \vdash_{[]} A\langle 0\rangle$, $\top \vdash_{x:A} A\langle 1\rangle(x)$, and for any map $h : m \longrightarrow n$ of finite sets

$$A\langle n\rangle(x_1, \ldots, x_n) \ \vdash_{\vec{x}} \ A\langle m\rangle(x_{h(1)}, \ldots, x_{h(m)})$$

(2) (**Functionality**) For each n-ary function symbol f in Σ-Fun

$$G_f(x_1, \ldots, x_n, y) \ \vdash_{\vec{x},y} \ A\langle n\rangle(x_1, \ldots, x_n),$$
$$A\langle n\rangle(x_1, \ldots, x_n) \ \vdash_{\vec{x}} \ (\exists! y : A) G_f(x_1, \ldots, x_n, y)$$

(3) (**Neighbourhood**) For a finite family of terms-in-context $\vec{x}.t_j$, $1 \leq j \leq n$

$$\bigwedge_{1\leq j\leq n} \phi_{t_j} \ \vdash_{\vec{x},\vec{y}} \ A\langle n\rangle(y_{t_1}, \ldots, y_{t_n})$$

(The i-structure axioms in (1) follow from the neighbourhood axiom, if we allow $n = 0$ and define the empty conjuction as '$\top$'.)

(4) For each axiom $\top \vdash_{\vec{x}} t = s$, $\mathrm{I}[\mathbb{T}]$ has an axiom

$$A\langle n\rangle(x_1, \ldots, x_n) \ \vdash_{\vec{x}} \ (\exists! y)\phi_t[\vec{x}, y/\vec{x}, y_t] \wedge \phi_s[\vec{x}, y/\vec{x}, y_s]$$

where n is the length of the context $\vec{x}$.

As pointed out in the introduction, the intuition here is that an IMAT A of $\mathbb{T}$ is a partial $\mathbb{T}$-algebra. The domain of definition is specified for each arity, rather than for each individual function symbol. Another noteworthy difference to the usual treatment of partial algebras is that variables-in-context are not totally defined terms; the domain of definition is always prescribed by the i-structure and the type of the context.

With the correct interpretation of variables terms can be constructed by the rules of composition of partial morphisms (i.e. the composition of their graphs as relations) yielding the i-structure as domains of definition according to the context (due to the neighbourhood axiom). Rather than cartesian logic a more suitable logical formalism for IMATs might be *partial Horn logic* as introduced in [PV07] (if the interpretation of variables can be adapted to our case).

In this thesis we will continue working with cartesian logic. We show that $\mathrm{I}[\mathbb{T}]$-models are the same as i-$O_{\mathbb{T}}$-algebras next.

Theorem 2.4.2. *Let $\mathbb{T}$ be an algebraic theory over a signature Σ and $O_{\mathbb{T}}$ its clone as in theorem* 1.4.1. *The categories $O_{\mathbb{T}}$ -IAlg(C) and* I$[\mathbb{T}]$ -Mod(C) *are isomorphic as categories over a finite-limit category C.*

Proof. (1) Let (A, α) be an i-$O_{\mathbb{T}}$-algebra over C. Define an I$[\Sigma]$-structure M by $MA = A\langle 1\rangle$, $MA\langle n\rangle$ as the subobject $i_n : A\langle n\rangle \rightarrowtail A\langle 1\rangle^n$, and $MG_f \rightarrowtail A^{n+1}$ as the graph of (the partial morphism) $\alpha_n([f(x_1, \ldots, x_n)])$ for $f \in \Sigma$-Fun of arity n:

$$A\langle n\rangle \rightarrowtail \xrightarrow{(i_n, \alpha_n([f(x_1,\ldots,x_n)]))} A^{n+1}$$

Recall that $O_{\mathbb{T}}(n)$ is the $\mathbb{T}$-algebra $T_\Sigma(n)/E_n$. We shall abbreviate $\alpha_n([f(x_1, \ldots, x_n)])$ by $\alpha_n([f])$. The i-structure axioms are satisfied in M as has been shown in lemma 2.1.14. Note that the functionality axiom (2) is equivalent to saying that MG_f and $MA\langle n\rangle$ are equal as subobjects of MA^n

$$\begin{tikzcd} MA\langle n\rangle \arrow[r, "\cong"] \arrow[d, tail, "i_n"'] & MG_f \arrow[d, tail] \\ MA^n & MA^n \times MA \arrow[l, "\mathrm{pr}_1"'] \end{tikzcd}$$

which is equivalent to saying that MG_f is (isomorphic to) the graph $(i_n, f) : MA\langle n\rangle \longrightarrow MA^n \times MA$ of a morphism $f : MA\langle n\rangle \longrightarrow MA$ (considered as a partial morphism). The sequents in axiom (2) are hence satisfied due to the definition of MG_f.

As regards axiom (3) and (4) note first that ϕ_t expresses the compositional structure of a Σ-term t in terms of the composition of the respective graphs. An induction over the structure of a term-in-context $\vec{x}.t$ shows that the interpretation of $x, y_t.\phi_t$ is the graph of $\alpha_n([t])$, where n is the length of the context $\vec{x}$.

If t is a variable x_j then ϕ_t is $(y_t = x_j) \wedge A\langle n\rangle(x_1, \ldots, x_n)$. The interpretation of $\vec{x}, y_t.\phi_t$ is the graph of the (restriction of the) projection $\mathrm{pr}_j \circ i_n : A\langle n\rangle \longrightarrow A$; but $\alpha_n([x_j]) = \mathrm{pr}_j \circ i_n$. By the composition axiom of definition 2.2.4, if t is $f(t_1, \ldots, t_m)$ then $\alpha_m([t])$ is $\alpha_m([f]) \circ (\alpha_n([t_1]), \ldots \alpha_n([t_m]))$. (Recall that the clone composition '$*$' in $O_{\mathbb{T}}$ is induced by substitution of terms.) The interpretation of the second clause of the definition of ϕ_t constructs $[\![\vec{x}, y_t.\phi_t]\!]$ as the (multi-) composition of the relations $[\![\vec{x}, y_{t_j}.\phi_{t_j}]\!]$, which are the graphs of $\alpha_n([t_j])$ by induction hypothesis. The composite relation $[\![\vec{x}, y_t.\phi_t]\!]$ is the graph of the composite $\alpha_m([f]) \circ (\alpha_n([t_1]), \ldots \alpha_n([t_m]))$, which is $\alpha_m([t])$ as asserted. (Note that the composite $\alpha_m([f]) \circ (\alpha_n([t_1]), \ldots \alpha_n([t_m]))$ is well-defined due to the neighbourhood axiom of definition 2.2.4, so $[\![\vec{x}, y_t.\phi_t]\!]$ is indeed a graph of a morphism.)

The interpretation $[\![\vec{x}, \vec{y}.\ \bigwedge_{1\leq j\leq n} \phi_{t_j}]\!]$ is the graph

$$MA\langle m\rangle \xrightarrow{(i_m,(\alpha_m([t_1]),\ldots,\alpha_m([t_n])))} MA^m \times MA^n$$

The neighbourhood axiom (3) is satisfied in M if and only if this graph factors through the subobject $1_{MA^m} \times i_n : MA^n \times MA\langle n\rangle \rightarrowtail MA^m \times MA^n$. But this is the case if and only if $(\alpha_m([t_1]), \ldots, \alpha_m([t_n]))$ factors through i_n, which it does due to the neighbourhood axiom of definition 2.2.4.
As regards axiom (4), if $\mathbb{T}$ has the axiom $\top \vdash_{\vec{x}} s = t$, then

$$A\langle n\rangle(x_1, \ldots, x_n) \vdash_{\vec{x}} (\exists! y)\phi_t[\vec{x}, y/\vec{x}, y_t] \wedge \phi_s[\vec{x}, y/\vec{x}, y_s]$$

is equivalent to saying that $[\![\vec{x}, y.\phi_t[\vec{x}, y/\vec{x}, y_t]]\!]$ and $[\![\vec{x}, y.\phi_t[\vec{x}, y/\vec{x}, y_t]]\!]$ are the graphs of the same morphism. Since $[s] = [t]$ by the construction of E_n, and hence $\alpha_n([s]) = \alpha_n([t])$, this is the case, indeed.

(2) Let M be an I[$\mathbb{T}$]-model in C. By lemma 2.1.14 and the i-structure axioms in (1) the family of subobjects

$$MA\langle n\rangle \xrightarrow{i_n} MA^n$$

constitutes an i-structure over MA. For each term $t \in T_\Sigma(n)$ (considered in the context $\vec{x} = x_1, \ldots, x_n$) we show that the formula-in-context $\vec{x}, y_t.\phi_t$ is I[$\mathbb{T}$]-provably functional. Indeed, the sequent $\phi_t \vdash_{\vec{x},y_t} A\langle n\rangle(x_1, \ldots, x_n)$ is satisfied in M due to the first clause of the definition of ϕ_t. We show the satisfaction of

$$A\langle n\rangle(x_1, \ldots, x_n) \vdash_{\vec{x}} (\exists! y_t)\phi_t$$

by induction over the structure of ϕ_t. Clearly, the sequent

$$A\langle n\rangle(x_1, \ldots, x_n) \vdash_{\vec{x}} (\exists! y_t)(y_t = x_j \wedge A\langle n\rangle(x_1, \ldots, x_n))$$

is satisfied in M. If t is $f(t_1, \ldots, t_m)$ then the induction hypothesis, the conjunction rule and Frobenius yield the satisfaction of the sequent

$$A\langle n\rangle(x_1, \ldots, x_n) \vdash_{\vec{x}} (\exists! y_{t_1}) \cdots (\exists! y_{t_m}) \bigwedge_{1\leq j\leq m} \phi_{t_j}$$

The neighbourhood axiom (3) states that

$$\bigwedge_{1\leq j\ m} \phi_{t_j} \vdash_{\vec{x},\vec{y}} A\langle m\rangle(y_{t_1}, \ldots, y_{t_m})$$

is satisfied in M. Together with the functionality axiom (2) we see that M satisfies

$$(\exists! y_{t_1}) \cdots (\exists! y_{t_m}) \bigwedge_{1\leq j\ m} \phi_{t_j} \vdash_{\vec{x}} (\exists! y_t)(\exists! y_{t_1}) \cdots (\exists! y_{t_m}) G_f(y_{t_1}, \ldots, y_{t_m}, y_t) \wedge \bigwedge_{1\leq j\ m} \phi_{t_j}$$

We conclude that the asserted sequent is satisfied after applying the cut rule.
Since ϕ_t has been shown functional, its interpretation $[\![\vec{x}, y_t.\phi_t]\!]$ is the graph of the morphism $MA\langle n\rangle \to MA$:

$$MA\langle n\rangle \xrightarrow{\cong} [\![\vec{x}, y_t.\phi_t]\!] \rightarrowtail MA^n \times MA \xrightarrow{\mathrm{pr}_2} MA$$

This defines a map

$$\alpha_n : T_\Sigma(n) \to C(MA\langle n\rangle, MA)$$

for each $n \in \mathbb{N}$, which maps the variables x_j to the projections $\mathrm{pr}_j \circ i_n$ as we have seen in part (1). The maps α_n thus satisfy the projection axiom of definition 2.2.4 for the clone $O_\emptyset$ (over Σ). (Note that for an n-ary function symbol $f \in \Sigma$-Fun the formulae-in-context $\vec{x}, y_t.G_f(x_1, \ldots, x_n, y_t)$ and $\vec{x}, y_t.\phi_t$ for $t = f(x_1, \ldots, x_n)$ are I[$\mathbb{T}$]-provably equivalent, and hence are both interpreted by the graph of the same morphism, of which MG_f is the graph.)
The interpretation $[\![\vec{x}, \vec{y}. \bigwedge_{1\leq j\leq n} \phi_{t_j}]\!]$ is the graph $(i_m, (\alpha_m(t_1), \ldots, \alpha_m(t_n)))$, and the neighbourhood axiom (3) says that $(\alpha_m(t_1), \ldots, \alpha_m(t_n))$ factors through i_m, as we have seen in part (1). The maps α_n thus satisfy the neighbourhood axiom of definition 2.2.4. We can apply this factorisation to show by an induction similar to the one in part (1) that the interpretation of $\phi_{t[\vec{s}/\vec{x}]}$ is the graph of the composite $\alpha_n(t) \circ (\alpha_m(s_1), \ldots, \alpha_m(s_n))$. This shows the composition axiom of definition 2.2.4 for α.
As we have noted in part (1) axiom (4) says that for any sequent $\top \vdash_{\vec{x}} s = t$ the interpretations of ϕ_t and ϕ_s are the graph of the same morphism, and hence $\alpha_n(s) = \alpha_n(t)$. Due to the composition axiom the map α_n thus respects the equivalence relation E_n, and hence factors through $T_\Sigma(n)/E_n = O_{\mathbb{T}}(n)$. We denote the resulting maps $O_{\mathbb{T}}(n) \to C(MA\langle n\rangle, MA)$ by α_n again. With this we have shown that (MA, α) is an i-$O_{\mathbb{T}}$-algebra.

(3) The constructions defined in part (1) and (2) are mutually inverse. Indeed, since i_n is monic, any morphism h such that the following diagram commutes

$$\begin{tikzcd} MA\langle n\rangle \arrow[rr, "h"] \arrow[dr, tail, "(i_n{,}f)"'] & & MA\langle n\rangle \arrow[dl, tail, "(i_n{,}g)"] \\ & MA^n \times MA & \end{tikzcd}$$

must be the identity morphism. This implies that when given an i-$O_{\mathbb{T}}$-algebra (A, α) to construct the corresponding $\mathrm{I}[\mathbb{T}]$-model, constructing an i-$O_{\mathbb{T}}$-algebra algebra out of it will yield (A, α) again, since the original α is reconstructed out of the graphs $[\![\vec{x}, y_t.\phi_t]\!]$. In the converse direction, since MG_f is the graph of the same morphism as the interpretation of $\phi_{f(x_1,\dots,x_n)}$, we recover the original $\mathrm{I}[\Sigma]$-structure M out of the i-$O_{\mathbb{T}}$-algebra (MA, α).

(4) Any i-$O_{\mathbb{T}}$-homomorphism $h : (A, \alpha) \to (A', \alpha')$ becomes an $\mathrm{I}[\Sigma]$-homomorphism of the corresponding $\mathrm{I}[\Sigma]$-structures M and M' (and vice versa). Indeed, h is an i-morphism and hence renders the subsequent diagram commutative

$$\begin{tikzcd} MA\langle n\rangle \arrow[r, tail] \arrow[d] & MA^n \arrow[d, "h^n"] \\ M'A\langle n\rangle \arrow[r, tail] & M'A^n \end{tikzcd}$$

Furthermore, since $h \circ \alpha_n = \alpha'_n \circ h^n$, and MG_f and $M'G_f$ are the graphs of $\alpha_n([f])$ and $\alpha'_n([f])$, respectively, the subsequent diagrams are rendered commutative as well

$$\begin{tikzcd} MG_f \arrow[r, tail] \arrow[d] & MA^n \times MA \arrow[d, "h^n \times h"] \\ M'G_f\langle n\rangle \arrow[r, tail] & M'A^n \times M'A \end{tikzcd}$$

Conversely, any $\mathrm{I}[\Sigma]$-homomorphism $h : MA \to M'A$ is an i-morphism. Since it renders commutative the preceding diagram for each G_f, it also makes the following diagrams commute for each Σ-term t [Joh02, lem. D1.2.9]

$$\begin{tikzcd} [\![\vec{x}, y_t.\phi_t]\!]_M \arrow[r, tail] \arrow[d] & MA^n \times MA \arrow[d, "h^n \times h"] \\ [\![\vec{x}, y_t.\phi_t]\!]_{M'} \arrow[r, tail] & M'A^n \times M'A \end{tikzcd}$$

But this is equivalent to saying that $h \circ \alpha_n(t) = \alpha'_n(t) \circ h^n$, for the interpretations of ϕ_t are the graphs of the respective morphisms $\alpha_n(t)$ and $\alpha'_n(t)$.
Together with part (3) we may conclude that the constructions defined in part (1) and (2) yield an isomorphism of $O_{\mathbb{T}}$ -IAlg(C) and I[$\mathbb{T}$] -Mod(C) as categories over C, as asserted.

$\square$

Remark 2.4.3. The category $\mathbb{T}$ -Mod(C) has a fully faithful embedding into I[$\mathbb{T}$] -Mod(C) as categories over C: a Σ-structure M defines an I[Σ]-structure M by defining $MA\langle n\rangle = MA^n$ and MG_f as the graph of Mf. Clearly, the I[Σ]-structure M is an I[$\mathbb{T}$]-model if the Σ-structure M is a $\mathbb{T}$-model. In fact, we just translate the function symbols of Σ into their graphs in this process and adapt the axioms accordingly. (See [Joh02, lem. D1.4.9].)

The isomorphism of theorem 2.4.2 maps the embedded $\mathbb{T}$-models to total i-$O_{\mathbb{T}}$-algebras and vice versa. In fact, it restricts to an isomorphism

$$\begin{tikzcd}
O_{\mathbb{T}}\text{-Alg}(C) \arrow[r, "\cong"] \arrow[d, tail] & \mathbb{T}\text{-Mod}(C) \arrow[d, tail] \\
O_{\mathbb{T}}\text{-IAlg}(C) \arrow[r, "\cong"] & \mathrm{I}[\mathbb{T}]\text{-Mod}(C)
\end{tikzcd}$$

The top isomorphism is the one constructed in proposition 1.3.18. Indeed, the construction in theorem 2.4.2 is just the construction in proposition 1.3.18 in the disguise of graphs instead of morphisms. In this sense theorem 2.4.2 extends the isomorphism of algebras of a Set-clone and models of its corresponding algebraic theory to its i-algebras and models of the i-sation of that theory.

Remark 2.4.4. The *functional approach to infinitesimalisation* of an algebraic theory $\mathbb{T}$ introduces the i-structure by adding new sorts. The function symbols of $\mathbb{T}$ can then be defined as (total) operations in I[$\mathbb{T}$]. However, the Σ-terms have to be defined by functional relations, still. Note that the functional approach is essentially an adapted version of the translation of relations into new sorts and function symbols (cf. [Joh02, lem. D1.4.9]).
The signature I[Σ] is defined as follows:

- For each $n \in \mathbb{N}$, I[Σ]-Sort has a sort $A\langle n\rangle$. ($A\langle 0\rangle$ is only introduced to avoid case distinctions and could be replaced by the empty context.)

- For each function symbol $f : A \cdots A \to A$ of arity n in Σ-Fun there is a function symbol $f : A\langle n\rangle \to A\langle 1\rangle$ in I[Σ]-Fun.

- For each $n \geq 1$ and $1 \leq j \leq n$, I[Σ]-Fun has a function symbol $i_{n,j} : A\langle n\rangle \to A\langle 1\rangle$.

- $\mathrm{I}[\Sigma]$ has no relation symbols.

For each Σ-term-in-context $\vec{x}.t$ there is a formula-in-context $x : A\langle n\rangle, y_t : A\langle 1\rangle.\phi_t$ over $\mathrm{I}[\Sigma]$ that is defined by recursion over the structure of t, where n is the length of the context $\vec{x} = x_1, \dots, x_n$.

- If t is a variable x_j, then ϕ_t is $y_t = i_{n,j}(x)$.
- If t is $f(t_1, \dots, t_m)$ then ϕ_t is

$$(\exists! x_f)(\exists! y_{t_1})\cdots(\exists! y_{t_m})(y_t = f(x_f)) \wedge \bigwedge_{1\le j\le m} ((i_{m,j}(x_f) = y_{t_j}) \wedge \phi_{t_j})$$

  where $x_f : A\langle m\rangle$ and $y_{t_j} : A\langle 1\rangle$. (Note that the use of the existential quantifiers for y_{t_j} is sound relative to the empty theory. The existential quantifier for x_f is sound relative to the axiom (1) below.)

The axioms of $\mathrm{I}[\mathbb{T}]$ are as follows:

(1) *I-structure*: For each $n \ge 1$ the $i_{n,j}$ are jointly monic

$$\bigwedge_{1\le j\le n} i_{n,j}(x) = i_{n,j}(z) \ \vdash_{x:A\langle n\rangle, z:A\langle n\rangle} \ x = z$$

and $\quad \top \vdash_{[]} (\exists! x : A\langle 0\rangle)\top$

(2) *Neighbourhood*: For a finite family of terms-in-context $\vec{x}.t_j$, $1 \le j \le n$ (where $\vec{x}$ is of length m, say)

$$\top \ \vdash_{x:A\langle m\rangle} \ (\exists! y : A\langle n\rangle) \bigwedge_{1\le j\le n} \phi_{t_j}[x, i_{n,j}(y)/x, y_{t_j}]$$

(The use of the existential quantifier is sound relative to (1))

(3) For each axiom $\top \vdash_{\vec{x}} t = s$, $\mathrm{I}[\mathbb{T}]$ has an axiom

$$\top \ \vdash_{x:A\langle n\rangle} \ (\exists! y : A\langle 1\rangle)\, \phi_t[x, y/x, y_t] \wedge \phi_s[x, y/x, y_s]$$

2.5 Infinitesimalisation of cartesian theories

If we have a K-algebra R in Set, then we can consider the i-sation of the algebraic theory $\mathbb{T}_{\mathcal{A}_R}$ of affine combinations over R. ($\mathbb{T}_{\mathcal{A}_R}$ is the algebraic theory corresponding to the clone of affine combinations $\mathcal{A}_R$ as in theorem 1.4.1.) Theorem 2.4.2 then shows that the category of $\mathrm{I}[\mathbb{T}_{\mathcal{A}_R}]$-models is equivalent to the category of i-affine spaces (over R). If R is a K-algebra object in a finite-limit category C, however, then we cannot construct an algebraic theory of affine combinations over R, in general. I-sation of algebraic theories thus cannot recapture all i-affine spaces.

Similarly, we have seen in remark 1.4.2 that categories of algebras of a clone O are full subcategories of models of a many-sorted algebraic theory. However, because it is many-sorted, the infinitesimalisation construction cannot be applied in its current form and has to be extended. Indeed, infinitesimalisation can be extended to the many-sorted case, and, in fact, to the class of cartesian theories.

Definition 2.5.1 (*i-sation of cartesian theories*)**.** Let $\mathbb{T}$ be a cartesian theory over the signature Σ. We propose the subsequent construction as the i-sation $\mathrm{I}[\mathbb{T}]$ of $\mathbb{T}$ extending the previously defined i-sation of (one-sorted) algebraic theories:

- $\mathrm{I}[\Sigma]$-Sort = Σ-Sort
- $\mathrm{I}[\Sigma]$ has no function symbols.
- Σ-Rel $\subset$ $\mathrm{I}[\Sigma]$-Rel
- For each list $A_1 \cdots A_n$ of sorts A_i in Σ and $n \in \mathbb{N}$, $\mathrm{I}[\Sigma]$ has an n-ary relation symbol

$$A\langle n\rangle \rightarrowtail A_1 \cdots A_n$$

- For each n-ary function symbol $f : A_1 \cdots A_n \to B$ in Σ-Fun, there is an $(n+1)$-ary relation symbol
$$G_f \rightarrowtail A_1 \cdots A_n B$$

For each Σ-term-in-context $\vec{x}.t$ of sort B there is a formula-in-context $\vec{x}, y_t.\phi_t$ over $\mathrm{I}[\Sigma]$ (with $y_t : B$) that is defined by recursion over the structure of t:

- If t is a variable x_j, then ϕ_t is

$$(y_t = x_j) \wedge A\langle n\rangle(x_1, \ldots, x_n),$$

where $A\langle n\rangle \rightarrowtail A_1 \cdots A_n$ and $A_1 \cdots A_n$ is the type of the context $\vec{x}$.

- If t is $f(t_1, \ldots, t_m)$ (where $t_j : A_j$, say) then ϕ_t is

$$(\exists! y_{t_1} : A_1) \cdots (\exists! y_{t_m} : A_n) G_f(y_{t_1}, \ldots, y_{t_n}, y_t) \wedge \bigwedge_{1 \le j \le m} \phi_{t_j}$$

The axioms of $\mathrm{I}[\mathbb{T}]$ are as follows:

(1) (**I-structure**) $\top \vdash_{[]} A\langle 0\rangle$ and $\top \vdash_{x:A} A\langle 1\rangle(x)$ for each sort A. Moreover, for any two lists of sorts $A_1 \cdots A_n$ and $B_1 \cdots B_m$, and a map $h : m \to n$ of finite sets such that $B_j = A_{h(j)}$

$$A\langle n\rangle(x_1, \ldots, x_n) \vdash_{x_1:A_1,\ldots,x_n:A_n} B\langle m\rangle(x_{h(1)}, \ldots, x_{h(m)})$$

(2) (**Functionality**) For each $f : A_1 \cdots A_n \to B$

$$G_f(x_1, \ldots, x_n, y) \vdash_{\vec{x},y:B} A\langle n\rangle(x_1, \ldots, x_n)$$
$$A\langle n\rangle(x_1, \ldots, x_n) \vdash_{\vec{x}} (\exists! y : B) G_f(x_1, \ldots, x_n, y)$$

where $A\langle n\rangle \rightarrowtail A_1 \cdots A_n$.

(3) (**Neighbourhood**) For a finite family of terms-in-context $\vec{x}.t_j$ (of sort B_j, say), $1 \le j \le n$

$$\bigwedge_{1 \le j \le n} \phi_{t_j} \vdash_{\vec{x},\vec{y}} B\langle n\rangle(y_{t_1}, \ldots, y_{t_n})$$

where $B\langle n\rangle \rightarrowtail B_1 \cdots B_n$.

(4) (**Relations**) For each relation $R \rightarrowtail A_1 \cdots A_n$ in Σ

$$R(x_1, \ldots, x_n) \vdash_{\vec{x}} A\langle n\rangle(x_1, \ldots, x_n)$$

where $A\langle n\rangle \rightarrowtail A_1 \cdots A_n$.

(5) Let $\sigma \in \mathbb{T}$ be $\phi \vdash_{\vec{x}} \psi$ with $\vec{x}$ of type $A_1 \cdots A_n$. Construct a sequent $\sigma_I \in \mathrm{I}[\mathbb{T}]$ out of σ as follows:

- Replace ϕ by $A\langle n\rangle(x_1, \ldots, x_n) \wedge \phi$, where $A\langle n\rangle$ is of type $A_1 \cdots A_n$.
- Replace each equation $s = t$ in σ by

$$(\exists! y)\phi_t[\vec{x}, y/\vec{x}, y_t] \wedge \phi_s[\vec{x}, y/\vec{x}, y_s]$$

- Replace each $R(t_1, \ldots, t_n)$ in σ by

$$(\exists! y_{t_1} : A_1) \cdots (\exists! y_{t_n} : A_n) \bigwedge_{1 \le j \le n} \phi_{t_j} \wedge R(y_{t_1}, \ldots y_{t_n})$$

  where $R \rightarrowtail A_1 \cdots A_n$.

2.5.1 A theory of infinitesimally affine spaces

With the i-sation of cartesian theories available we can consider also non-algebraic presentations of the theory of affine spaces. One such presentation, which is popular in affine geometry, is to define affine spaces as torsors of modules (over commutative rings).

The signature has three sorts R, V and A standing for the ring R, R-module V and the affine space A. The function symbols are the usual ring operations and constants for R, the group operations and constant 0 for V, the operation $RV \longrightarrow V$, and an operation $\alpha : VA \longrightarrow A$. The axioms are the usual algebraic commutative ring axioms, algebraic axioms of an R-module and that of an action of an abelian group on a set for α. So far this is a many-sorted algebraic theory. The non-algebraic axiom states that α is a simply transitive action[3]

$$\top \ \vdash_{P:A,Q:A} \ (\exists! v : V)\alpha(P, v) = Q$$

An infinitesimalisation of this theory would give us an i-ring R, an abelian i-group V, an i-structure over A and further mixed i-neighbourhoods like $\langle R \cdots RV \rangle \rightarrowtail R \cdots RV$ and $\langle V \cdots VA \rangle \rightarrowtail V \cdots VA$, for example. The axioms would make sure that all operations are compatible with the i-structures. There are models of this theory that we do not want to consider as i-affine spaces though. For example, we would want the i-ring to be total, and also the action $RV \longrightarrow V$ to be total (as it would be, if we had just considered the i-sation of the one-sorted algebraic theory of an R-module over a fixed (Set-)ring R). We will not investigate this presentation any further here. However, we will see i-R-modules in the application of i-algebra to naive SDG in chapter 3.3, where they are called *i-vector spaces*.

There is also an *essentially algebraic theory* of affine spaces $\mathbb{A}$. (Cf. [AR94, def. 3.34] for the definition of an essentially algebraic theory.) Since it is a theory of affine combinations it is closer to our original definition of i-affine spaces and hence easier to compare.

The signature of $\mathbb{A}$ has two sorts R (for the ring) and A (for the affine space). The operations are actions of R-linear combinations on A, but only defined when they are affine combinations,

[3]This theory does have a presentation as a many-sorted algebraic theory: we introduce a further operation $- : AA \longrightarrow V$ and the axioms $\top \vdash_{P,Q} \alpha(P, Q - P) = Q$, $\top \vdash_{P,v} \alpha(P, v) - P = v$.

i.e. when the coefficients sum up to 1. Besides the ring axioms the axioms are the equations induced by the clone $\mathcal{A}$ of affine combinations like in part (2) of the proof of theorem 1.4.1. More formally and in the language of cartesian logic:

Definition 2.5.2 (*Theory of affine spaces*)**.** The theory $\mathbb{A}$ of **affine spaces** over the signature Σ is defined as follows.

- Σ has two sorts: R and A
- Σ has function symbols

$$+ : RR \longrightarrow R, \quad \cdot : RR \longrightarrow R, \quad - : RR \longrightarrow R, \quad 0 : \longrightarrow R, \quad 1 : \longrightarrow R$$

- Σ has a relation symbol

$$\Lambda_n \rightarrowtail \underbrace{R \cdots R}_{n} \underbrace{A \cdots A}_{n+1}$$

  for each $n \geq 1$ of arity $2n + 1$.

The axioms of $\mathbb{A}$ are

(1) The algebraic axioms of a commutative ring with 1; i.e. the algebraic axioms of an abelian group for '+', '−', '0' and commutative monoid for '·', '1' together with the distributivity laws. (From now on all rings are assumed to be commutative with 1, unless stated otherwise.)

(2) *Functionality*: For every $n \geq 1$

$$\Lambda_n(\lambda_1, \ldots, \lambda_n, P_1, \ldots, P_n, Q) \vdash_{\vec{\lambda}, \vec{P}, Q} \sum_{i=1}^{n} \lambda_i = 1$$

$$\sum_{i=1}^{n} \lambda_i = 1 \vdash_{\vec{\lambda}, \vec{P}} (\exists! Q : A) \Lambda_n(\lambda_1, \ldots, \lambda_n, P_1, \ldots, P_n, Q)$$

(3) *Associativity*: For each $n, m \geq 1$

$$(\exists! \vec{Q}) \bigwedge_{1 \leq j \leq n} \Lambda_m(\vec{\lambda}_j, \vec{P}, Q_j) \wedge \Lambda_n(\vec{\mu}, \vec{Q}, Y) \dashv\vdash_{\vec{\mu}, \vec{\lambda}_1, \ldots, \vec{\lambda}_n, \vec{P}, Y} \Lambda_m\Big(\sum_{j=1}^{n} \mu_j \vec{\lambda}_j, \vec{P}, Y\Big)$$

where we have used the vector notation from chapter 1.3.5, and '$\dashv\vdash$' means provable equivalence.

(4) *Projection*: For every $n \geq 1$ and $1 \leq j \leq n$

$$\Lambda_n(0, \ldots, 0, \underset{j}{1}, 0, \ldots, 0, P_1, \ldots, P_n, Q) \vdash_{\vec{P},Q} Q = P_j$$

An $\mathbb{A}$-model M in a category C with finite limits is a ring object MR and an object MA together with morphisms

$$\alpha_n : \mathcal{A}_{MR}(n) \times MA^n \longrightarrow MA, \qquad n \geq 1,$$

for which $M\Lambda_n$ is the respective graph (of the partial morphism) in C, and $\mathcal{A}_{MR}(n)$ is defined as in corollary 1.3.23 (for $K = \mathbb{Z}$). The projection and associativity axioms state that α_n satisfies the associativity and projection axioms of an $\mathcal{A}_{MR}$-algebra, apart from the ones involving α_0 (which is not defined in this approach, since the theory of affine spaces has no constants). In fact, the associativity and projection axioms are equivalent to their respective counterparts in definition 1.3.5; this follows from a similar argument as in the proof of theorem 2.4.2, where the (multi-)composition in the neighbourhood axiom is represented by the (multi-)composition of the corresponding graphs. Every $\mathcal{A}_R$-algebra (A, α) thus yields an $\mathbb{A}$-model M over R (meaning $MR = R$), and like in step (4) of theorem 2.4.2 we see that every $\mathcal{A}_R$-homomorphism h becomes a Σ-homomorphism h (when h_R is taken to be the identity morphism). Hence for each ring object R in C there is an embedding of categories over C

$$\begin{tikzcd} \mathcal{A}_R\text{-Alg}(C) \arrow[rr, tail] \arrow[dr, "U"'] & & \mathbb{A}\text{-Mod}(C) \arrow[dl, "U_A"] \\ & C & \end{tikzcd}$$

(U_A maps an $\mathbb{A}$-model M to MA and a Σ-homomorphism h to h_A.) In a distributive category C, and for a non-trivial ring object R, the embedding is fully faithful; it is an isomorphism of $\mathcal{A}_R$-Alg(C) and $\mathbb{A}_R$-Alg(C), the full subcategory of $\mathbb{A}$-models over the ring object R. For a general finite-limit category C, or a ring that is not non-trivial (which does not necessarily imply that it is the trivial ring), the embedding into $\mathbb{A}_R$-Alg(C) might neither be full, nor surjective on objects. (See remark 1.3.25. For example, if R is the trivial ring 1 in Set, then 1 and $\emptyset$ are $\mathbb{A}$-models, but only 1 is an $\mathcal{A}_R$-algebra.) However, the embedding can be fully faithful, even if the category is not distributive. This is the case for the universal K-algebra in the syntactic category $C_{\mathbb{T}}$ of the (one-sorted) algebraic theory of K-algebras, for example.

The i-sation of $\mathbb{A}$ is again a too general theory for our purposes. We thus define the theory i$\mathbb{A}$ of i-affine spaces (over an arbitrary ring) ‘by hand’, and only use the i-sation construction as a guide. We want the ring R to be total, hence we only need to introduce an i-structure for A. The mixed i-neighbourhoods can also be avoided, as well as replacing the ring operations by their graphs. This also avoids translating terms into formulae, for the action of affine combinations is given by the graphs Λ_n already, and the only Σ-terms are formed from the ring operations only. We only need to add the i-structure and neighbourhood axiom, and add the sequents for the relations Λ_n, which can be included in the functionality axioms of $\mathbb{A}$. As for the translation of the axioms, we only need to replace the antecedent in each sequent by its conjunction with the respective $A\langle n\rangle$. The only axiom where this is needed is the second sequent of the functionality axiom.

Definition 2.5.3 (*Theory of i-affine spaces*)**.** The cartesian theory $\mathrm{i}\mathbb{A}$ of i-affine spaces has the signature $\mathrm{i}\Sigma$ given by

- $\mathrm{i}\Sigma$-Sort $= \{R, A\}$
- $\mathrm{i}\Sigma$ has function symbols

$$+ : RR \to R, \quad \cdot : RR \to R, \quad - : RR \to R, \quad 0 : \to R, \quad 1 : \to R$$

- $\mathrm{i}\Sigma$ has a relation symbol

$$\Lambda_n \rightarrowtail \underbrace{R \cdots R}_{n} \underbrace{A \cdots A}_{n+1}$$

  for each $n \geq 1$ of arity $2n + 1$.

- For each $n \in \mathbb{N}$, $\mathrm{i}\Sigma$ has an n-ary relation symbol

$$A\langle n\rangle \rightarrowtail A \cdots A$$

The axioms of $\mathrm{i}\mathbb{A}$ are as follows:

(1) The (algebraic) axioms of a commutative ring R with 1

(2) (**I-structure**) $\top \vdash_{[]} A\langle 0\rangle$, $\top \vdash_{x:A} A\langle 1\rangle(x)$, and for any map $h : m \to n$ of finite sets

$$A\langle n\rangle(x_1, \ldots, x_n) \vdash_{\vec{x}} A\langle m\rangle(x_{h(1)}, \ldots, x_{h(m)})$$

(3) (**Functionality**) For every $n \geq 1$

$$\Lambda_n(\lambda_1, \ldots, \lambda_n, P_1, \ldots, P_n, Q) \vdash_{\vec{\lambda}, \vec{P}, Q} \big(\sum_{i=1}^{n} \lambda_i = 1\big) \wedge A\langle n\rangle(P_1, \ldots, P_n)$$

$$\big(\sum_{i=1}^{n} \lambda_i = 1\big) \wedge A\langle n\rangle(P_1, \ldots, P_n) \vdash_{\vec{\lambda}, \vec{P}} (\exists! Q : A)\Lambda_n(\lambda_1, \ldots, \lambda_n, P_1, \ldots, P_n, Q)$$

(4) (**Neighbourhood**) For each $m, n \geq 1$

$$\bigwedge_{1 \leq j \leq n} \Lambda_m(\lambda_1^j, \ldots, \lambda_m^j, P_1, \ldots, P_m, Q_j) \vdash_{\vec{\lambda}^1, \ldots, \vec{\lambda}^n, \vec{P}, \vec{Q}} A\langle n\rangle(Q_1, \ldots, Q_n)$$

(5) (**Associativity**) For each $n, m \geq 1$

$$(\exists! \vec{Q}) \bigwedge_{1 \leq j \leq n} \Lambda_m(\vec{\lambda}_i, \vec{P}, Q_j) \wedge \Lambda_n(\vec{\mu}, \vec{Q}, Y) \dashv\vdash_{\vec{\mu}, \vec{\lambda}_1, \ldots, \vec{\lambda}_n, \vec{P}, Y} \Lambda_m\big(\sum_{j=1}^{n} \mu_j \vec{\lambda}_j, \vec{P}, Y\big)$$

(6) (**Projection**) For every $n \geq 1$ and $1 \leq j \leq n$

$$\Lambda_n(0, \ldots, 0, \underset{j}{1}, 0, \ldots, 0, P_1, \ldots, P_n, Q) \vdash_{\vec{P}, Q} Q = P_j$$

An i$\mathbb{A}$-model M in a finite-limit category C is a ring object MR together with an object MA and morphisms

$$\alpha_n : \mathcal{A}_{MR}(n) \times MA\langle n\rangle \to MA, \qquad n \geq 1,$$

which is almost an i-$\mathcal{A}_{MR}$-algebra, if not for the missing morphism α_0. Indeed, the α_n are recovered from the graphs of partial morphisms Λ_n, and the neighbourhood axiom gives us the required factorisation in definition 2.2.1(1); in fact, it is equivalent to it for the non-zero-case. By defining $M\Lambda_n$ as the graph of (the partial morphism) α_n an i-$\mathcal{A}_R$-algebra (A, α) yields an i$\mathbb{A}$-model over R, and i-$\mathcal{A}_R$-homomorphisms become iΣ-homomorphisms. The proofs are similar to the ones given in theorem 2.4.2, and extend the comparison of $\mathbb{A}$-models over R and $\mathcal{A}_R$-algebras. We summarise this in the subsequent theorem.

Theorem 2.5.4. *Let R be a ring object in a finite-limit category C. There is an embedding of categories over C*

$$\begin{tikzcd} \mathcal{A}_R\text{-IAlg}(C) \arrow[rr, tail] \arrow[dr, "U"'] & & i\mathbb{A}\text{-Mod}(C) \arrow[dl, "U_A"] \\ & C & \end{tikzcd}$$

For a distributive category C and a non-trivial ring object R the embedding is fully faithful. It induces an isomorphism of $\mathcal{A}_R$-IAlg(C) and $i\mathbb{A}_R$-Alg(C) as categories over C, where the latter denotes the full subcategory of $i\mathbb{A}$-models M with $MR = R$.

Remark 2.5.5. As in the case of affine spaces, the universal K-algebra in the syntactic category $C_{\mathbb{T}}$ of the (one-sorted) algebraic theory $\mathbb{T}$ of K-algebras induces a fully faithful embedding, but $C_{\mathbb{T}}$ is not distributive.

2.5.2 A theory of infinitesimal clone algebras

Another application of the i-sation of cartesian theories is the i-sation of the theory $\mathbb{C}$ of clone algebras defined in remark 1.4.2. As for i-affine spaces we shall use the infinitesimalisation construction as a guide only, and define the theory of infinitesimal clone algebras while introducing only the one necessary i-structure only.

Definition 2.5.6 (*Theory of i-clone algebras*)**.** Let Σ be the signature of the theory $\mathbb{C}$ of clone algebras. The cartesian theory i$\mathbb{C}$ of **infinitesimal clone algebras** over the signature iΣ is defined as follows:

- iΣ-Sort = Σ-Sort
- iΣ has a function symbol

  $$*_{nm} : O(n)O(m)\cdots O(m) \to O(m)$$

  of arity $n+1$ for every pair $(n,m) \in \mathbb{N}^2$ with $n \geq 1$, a unary function symbol

  $$*_{0m} : O(0) \to O(m)$$

  for each $m \in \mathbb{N}$, constants

  $$\pi^n_j : \to O(n)$$

- $\mathrm{i}\Sigma$ has a relation symbol
$$G_n \rightarrowtail O(n)\underbrace{A\cdots A}_{n+1}$$
of arity $n+2$ for each $n \in \mathbb{N}$.
- For each $n \in \mathbb{N}$, $\mathrm{i}\Sigma$ has an n-ary relation symbol
$$A\langle n\rangle \rightarrowtail A\cdots A$$

The axioms of $\mathrm{i}\mathbb{C}$ are as follows:

(1) The clone axioms of $\mathbb{C}$.

(2) (**I-structure**) $\top \vdash_{[]} A\langle 0\rangle$, $\top \vdash_{x:A} A\langle 1\rangle(x)$, and for any map $h : m \longrightarrow n$ of finite sets
$$A\langle n\rangle(x_1,\ldots,x_n) \;\vdash_{\vec{x}}\; A\langle m\rangle(x_{h(1)},\ldots,x_{h(m)})$$

(3) (**Functionality**) For every $n \in \mathbb{N}$
$$G_n(\sigma,x_1,\ldots,x_n,y) \;\vdash_{\sigma,\vec{x},y}\; A\langle n\rangle(x_1,\ldots,x_n)$$
$$A\langle n\rangle(x_1,\ldots,x_n) \;\vdash_{\vec{x}}\; (\exists! y : A)G_n(\sigma,x_1,\ldots,x_n,y)$$

(4) (**Neighbourhood**) For each $m,n \in \mathbb{N}$, $n \geq 1$
$$\bigwedge_{1\leq j\leq n} G_m(\sigma_j,x_1,\ldots,x_m,y_j) \;\vdash_{\vec{\sigma},\vec{x},\vec{y}}\; A\langle n\rangle(y_1,\ldots,y_n)$$
(In the case $m=0$ the context $\vec{x}$ is the empty context.)

(5) (**Associativity**) For each $n,m \in \mathbb{N}$
$$(\exists!\vec{y}) \bigwedge_{1\leq j\leq n} G_m(t_j,\vec{x},y_j) \wedge G_n(\sigma,\vec{y},z) \dashv\vdash_{\sigma,\vec{t},\vec{x},z} G_m(\sigma *_{nm} (t_1,\ldots,t_n),\vec{x},z)$$
(In the case $m=0$ the context $\vec{x}$ is the empty context. For $n=0$ the sequents become $G_0(\sigma,z) \dashv\vdash_{\sigma,\vec{x},z} G_m(*_{0m}(\sigma),\vec{x},z)$.)

(6) (**Projection**) For every $n \geq 1$ and $1 \leq j \leq n$
$$G_n(\pi^n_j,x_1,\ldots,x_n,y) \;\vdash_{\vec{x},y}\; y = x_j$$

A model M of i$\mathbb{C}$ in a finite-limit category C is a clone MO, an object MA with an i-structure $MA\langle - \rangle$ over it, and morphisms

$$\alpha_n : MO(n) \times MA\langle n \rangle \to MA, \qquad n \in \mathbb{N},$$

of which $MG_n \rightarrowtail MO(n) \times MA^n \times MA$ is the graph (of a partial morphism). With similar arguments as in the proof of theorem 2.4.2 we can see that the neighbourhood, associativity and projection axioms are equivalent to their respective counterparts in definition 2.2.1. (As in the case of $\mathbb{A}$ and i$\mathbb{A}$ the equality of the composite morphisms is just expressed as the equality of the composites of the respective graphs.) An i$\mathbb{C}$-model is thus an i-MO-algebra in C, and every i-O-algebra (A, α) in C yields an i$\mathbb{C}$-model M over O (i.e. $MO = O$), if we define MG_n as the graphs of the morphisms α_n. An i-O-algebra homomorphism h becomes an iΣ-homomorphism h, if we set $h_A = h$ and $h_{O(n)} = 1_{O(n)}$. The fully faithful embedding in remark 1.4.2 thus extends to:

Theorem 2.5.7. *Let O be a clone in a finite-limit category C, and let $\mathbb{C}$ be the* (*many sorted*) *algebraic theory of clone algebras as defined in remark* 1.4.2. *There is a fully faithful embedding of categories over C*

$$\begin{tikzcd} O\text{-IAlg}(C) \arrow[rr, tail] \arrow[dr, "U"'] & & i\mathbb{C}\text{-Mod}(C) \arrow[dl, "U_A"] \\ & C & \end{tikzcd}$$

where U_A maps an i$\mathbb{C}$-model M to MA and an iΣ-homomorphism h to h_A. ()*The theory*

The i-sation construction can be extended to all first-order theories and hence makes it possible to define infinitesimal models for such theories. As we have seen in the examples of the theory of affine spaces and i-clone algebras, however, the formally infinitesimalised theories would be too general. This is why the proposed i-sation of cartesian theories, unlike the i-sation of (one-sorted) algebraic theories, served merely as a guide in both of our examples. A more refined notion of i-sation of theories is needed; one which takes the sorts that are to be replaced by i-structures as parameters, and only makes a minimum of necessary changes to the signature and axioms.

Having a notion of infinitesimal model of a first-order theory might be interesting for applications to SDG. It allows us to define spaces that are modeled infinitesimally on geometries different from affine geometry; like *projective geometry*, or *Tarski's elementary Euclidean geometry*, for instance. Of course, at this stage it is neither clear whether there are non-trivial i-models of such theories in SDG, nor if such spaces have any geometric significance.

Unfortunately, we won't be able to investigate these questions here. The i-sation construction has been discovered by the author too late to be able to explore its implications any further in this thesis. It deserves more study.

2.6 Properties of categories of infinitesimal models

This section is devoted to the study of the properties of the category of IMATs over a Grothedieck topos $\mathcal{S}$. In particular, we shall study some lifting properties of limits and colimits of the forgetful functor

$$U : O\text{-IAlg}(\mathcal{S}) \to \mathcal{S}$$

Even though some of the properties we discuss hold for more general categories than Grothendieck toposes, the latter are the most important for applications. Note that since a Grothendieck topos is an infinitary distributive category the various notions of IMATs we have introduced in the previous sections coincide (apart from i-affine spaces that can be defined either as i-$\mathcal{A}_R$-algebras or $i\mathbb{A}_R$-models, which coincide only if the ring R is non-trivial).

Theorem 2.6.1. *Let $\mathcal{S}$ be a Grothendieck-topos and O a clone in $\mathcal{S}$. The category $O\text{-IAlg}(\mathcal{S})$ is locally presentable* [4].

Proof. We make use of the fully faithful embedding $O\text{-IAlg}(\mathcal{S}) \rightarrowtail i\mathbb{C}\text{-Mod}(\mathcal{S})$ of theorem 2.5.7 to show local presentability. Since $\mathcal{S}$ is locally presentable ([Bor94b, prop. 3.4.16] or [Joh02, cor. D2.3.7]), the category $\mathrm{i}\mathbb{C}\text{-Mod}(\mathcal{S})$ is locally presentable. This follows from the fact that a category of models of a limit-sketch in a locally presentable category is locally presentable [AR94, rem. 2.63], and that $\mathrm{i}\mathbb{C}\text{-Mod}(\mathcal{S})$ is equivalent to the category of models of a limit sketch [Joh02, example D2.1.4(d)].

The forgetful functor $U_{cl} : i\mathbb{C}\text{-Mod}(\mathcal{S}) \to \mathrm{clone}(\mathcal{S})$ is a cloven (Grothendieck) fibration [5]. (A cleavage is given by 'precomposing with clone homomorphisms' as in proposition 2.2.6(1).) The category of $\mathcal{S}$-models of the many-sorted algebraic theory $\mathbb{T}$ of clones and the category $\mathrm{clone}(\mathcal{S})$ are locally presentable (see remark 1.3.11). The forgetful functor U_O preserves filtered colimits and small limits. This can be seen with the help of the two syntactic categories $C_{i\mathbb{C}}$ and $C_{\mathbb{T}}$ as follows. By passing from $\mathrm{i}\mathbb{C}\text{-Mod}(\mathcal{S})$ and $\mathrm{clone}(\mathcal{S})$ to the equivalent categories of finite-limit preserving functors $C_{i\mathbb{C}} \to \mathcal{S}$ and $C_{\mathbb{T}} \to \mathcal{S}$, the forgetful functor U_{cl} becomes F_O^* for the functor $F_O : C_{\mathbb{T}} \to C_{i\mathbb{C}}$, which is induced by the clone O of the universal $i\mathbb{C}$-model in $C_{i\mathbb{C}}$.

[4] The definitions of a *locally presentable category*, *accessible category* and *accessible functor* can be found in [AR94].

[5] For the definition of a *fibration* and *cleavage* see [Joh02, def. B1.3.4].

Since limits and filtered colimits of finite-limit preserving functors into a Grothendieck topos are computed pointwise [Bor94b, prop. 3.4.5], they are preserved by U_{cl}. In particular, U_{cl} is an accessible functor. The U_{cl}-fibre over the clone O is isomorphic to O-IAlg($\mathcal{S}$) and we obtain a pullback diagram of categories

$$\begin{tikzcd} O\text{-IAlg}(\mathcal{S}) \arrow[r, tail] \arrow[d] & i\mathbb{C}\text{-Mod}(\mathcal{S}) \arrow[d, "U_{cl}"] \\ 1 \arrow[r, "O"] & \mathrm{clone}(\mathcal{S}) \end{tikzcd}$$

The functor O and U_{cl} are accessible and preserve small limits. The category O-IAlg($\mathcal{S}$) is thus accessible ([JS93, thm. 1], [AR94, thm. 2.77]), complete, and hence locally presentable [Joh02, prop. D2.3.4].

□

Corollary 2.6.2. O-IAlg($\mathcal{S}$) *is complete, cocomplete, well-powered and well-copowered.*

Proof. [AR94, rem. 1.56] □

Remarks 2.6.3.

(a) The same type of argument also shows IAff($\mathcal{S}$) locally presentable, when it is considered as the category i$\mathbb{A}_R$-Mod($\mathcal{S}$) for the respective (commutative) ring R, instead of the category $\mathcal{A}_R$-IAlg($\mathcal{S}$). Indeed, i$\mathbb{A}_R$-Mod($\mathcal{S}$) is the U_R-fibre over R for the forgetful functor $U_R : i\mathbb{A}\text{-Mod}(\mathcal{S}) \to \mathrm{CRng}(\mathcal{S})$, which is a cloven fibration, accessible and preserves small limits. (A cleavage is given by the 'restriction of scalars', which is a similar construction as for modules over rings.)

(b) If O comes from a Set-clone O', i.e. $O \cong \gamma^* O'$ where γ^* is the inverse-image functor of the (up to natural isomorphism) unique geometric functor $\mathcal{S} \to \mathrm{Set}$, then O-IAlg($\mathcal{S}$) is isomorphic to I[$\mathbb{T}_{O'}$]-Mod($\mathcal{S}$) as categories over $\mathcal{S}$. This follows from theorem 1.4.1(3), proposition 2.2.6, theorem 2.4.2 and remark 1.3.20(b). (Recall that γ^* is the finite-limit preserving functor induced by taking copowers of the terminal object in $\mathcal{S}$.) The local presentability of O-IAlg($\mathcal{S}$) can thus be concluded immediately.

We now turn to studying limits. As it is typical for categories of structures the forgetful functor lifts limits of the base category.

Theorem 2.6.4. *The forgetful functor* $U : O\text{-IAlg}(\mathcal{S}) \to \mathcal{S}$ *lifts small limits uniquely*; *that is, for every small diagram* $D : J \to O\text{-IAlg}(\mathcal{S})$ *and limiting cone* λ *of* $U \circ D$*, there is a unique limiting cone* μ *of* D *such that* $U\mu = \lambda$*. (See also [AHS05, def.* 13.17*].)*

Proof. Since the functor U has the property that an isomorphism h is the identity morphism if Uh is the identity morphism, all the limits U lifts it lifts uniquely. (See also [AHS05, prop. 13.21]. Note that U does neither reflect identities nor isomorphisms, in general.)

To see that U lifts limits we use again that $\mathbb{O}$-IAlg$(\mathcal{S})$ has a fully faithful embedding into the category i$\mathbb{C}$-Mod$(\mathcal{S})$ as categories over $\mathcal{S}$. Furthermore, the proof of theorem 2.6.1 shows that $\mathbb{O}$-IAlg$(\mathcal{S})$ is isomorphic to a limit-closed subcategory, so the embedding reflects limits. It is thus sufficent to show that the forgetful functor $U_A : i\mathbb{C}\text{-Mod}(\mathcal{S}) \to \mathcal{S}$ lifts limits. Since an equivalence of categories reflects limits, we can consider the equivalent category of finite-limit preserving functors $C_{i\mathbb{C}} \to \mathcal{S}$ (and natural transformations). The forgetful functor U_A becomes the evaluation functor ev_A at $A = \{(x : A).\top\}$. The limits of finite-limit preserving functors are computed pointwise, which shows that ev_A lifts limits. $\square$

Remarks 2.6.5.

(a) Let (A_j, α_j), $j \in I$ be a small family of i-$\mathbb{O}$-algebras in $\mathcal{S}$. The i-structure on the product $\prod_{j\in I} A_j$ is given by the **product i-structure**, which is defined by the respective product relations; that is $(\prod_{j\in I} A_j)\langle n\rangle$ is

$$\textstyle\prod_{j\in I}(A_j\langle n\rangle) \xrightarrow{\prod_{j\in I} i_{j,n}} \prod_{j\in I} A_j^n \xrightarrow{\cong} \left(\prod_{j\in I} A_j\right)^n$$

where $i_{j,n} : A_j\langle n\rangle \rightarrowtail A_j^n$ are the monomorphisms of the respective i-structure over A_j. The action α of $\mathbb{O}$ on the product i-structure is given componentwise

$$\begin{array}{rcl}\alpha_n : \mathbb{O}(n)\times\left(\prod_{j\in I} A_j\right)\langle n\rangle & \xrightarrow{\Delta_I\times 1_{(\prod_{j\in I} A_j)}} & \prod_{j\in I}\mathbb{O}(n) \times \prod_{j\in I} A_j\langle n\rangle \\ & & \Big\downarrow \cong \\ & & \prod_{j\in I}\mathbb{O}(n)\times A_j\langle n\rangle \xrightarrow{\prod_{j\in I}\alpha_{j,n}} \prod_{j\in I} A_j\end{array}$$

(b) U does not reflect small limits. A counterexample is provided by considering i-affine spaces in the syntactic category $C_{\mathbb{T}}$ of K-algebras. The section $Af : C_{\mathbb{T}} \to \mathrm{IAff}(C_{\mathbb{T}})$ of the forgetful functor U in theorem 2.3.1 does not preserve products, hence U cannot reflect products.

This is because the nil-square i-structure on A^2 is not the product i-structure. Indeed, we can find a K-algebra object, which does not satisfy the sequent

$$(x^2 = 0) \wedge (y^2 = 0) \vdash_{x:A,y:A} xy = 0.$$

(The sequent states that if $(0,0)$ and (x,y) are 1-neighbours in the product i-structure on A^2, then they are also 1-neighbours in the nil-square i-structure of the product.) For example, the K-algebra $R = y(A)$ given by the Yoneda embedding $y : C_{\mathbb{T}} \to [C_{\mathbb{T}}^{op}, \mathrm{Set}]$ cannot satisfy the above sequent. Indeed, R satisfies the *Kock-Lawvere axiom* [Koc06, cor. III.1.3], and hence the cancellation rule

$$(d^2 = 0) \wedge (dx = 0) \ \vdash_{d:R,x:R} \ x = 0$$

When combining this with the sequent above, we obtain $d^2 = 0 \ \vdash_{d:R} \ d = 0$, which together with the the Kock-Lawvere axiom would yield the contradiction that R is a trivial ring. Incidentally, since $\mathcal{S} = [C_{\mathbb{T}}^{op}, \mathrm{Set}]$ is a Grothendieck topos and because of corollary 2.3.3 (applied to y) we see that the forgetful functor $U : \mathrm{IAff}(\mathcal{S}) \to \mathcal{S}$ does not reflect products.

There is a geometric reason behind the fact that the nil-square i-structure of a product space is not the product i-structure, in general. Unlike for the nil-square i-structure, in the product i-structure on $X \times Y$ the components P and Q of a point (P,Q) are independent; there is no infinitesimal cohesion between P and Q. However, product spaces also model the geometric concept of a space having extension in different (independent) dimensions. The product i-structure on R^n is thus unsuitable for geometry, for it does not capture adequately the infinitesimal cohesion needed for geometry.

Next we study which colimits are lifted by U. Colimits are more intricate than limits, since it is not the case that U lifts all (small) colimits, in general. Also, colimits are more interesting to applications in SDG, for they correspond to gluing constructions of spaces.

Proposition 2.6.6. *U lifts filtered colimits uniquely.*

Proof. The proof is essentially the same as for the case of small limits. It is sufficient to show that U lifts filtered colimits. We mentioned before that filtered colimits of finite-limit preserving functors $C_{i\mathbb{C}} \to \mathcal{S}$ are computed pointwise. This shows that the evaluation functor ev_A defined on the the category of finite-limit preserving functors $C_{i\mathbb{C}} \to \mathcal{S}$ lifts filtered colimits. Since equivalences reflect colimits the forgetful functor $U_A : i\mathbb{C}\text{-Mod}(\mathcal{S}) \to \mathcal{S}$ lifts filtered colimits. For the same reason the forgetful functor $U_{cl} : i\mathbb{C}\text{-Mod}(\mathcal{S}) \to \mathcal{S}$ lifts filtered colimits, so the U_{cl}-fibre over O is closed under filtered colimits in $i\mathbb{C}\text{-Mod}(\mathcal{S})$. The embedding $O\text{-IAlg}(C) \rightarrowtail i\mathbb{C}\text{-Mod}(\mathcal{S})$ thus reflects filtered colimits, and hence U must lift them. □

Corollary 2.6.7 (Free i-O-algebras)**.** *U has a left adjoint.*

Proof. Since U lifts filtered colimits and small limits, it also preserves them. As a small-limit preserving, accessible functor, it has a left adjoint by the adjoint functor theorem for locally presentable categories [AR94, 1.66]. □

Remark 2.6.8 (*Discrete i-affine spaces*)**.** In the case of i-affine spaces over a non-trivial ring R we can describe the left adjoint of U explicitly: it is (a lift of) the discrete i-structure functor Δ of proposition 2.1.2.

To see this note first that for any clone O with $O(0) \cong \emptyset$ (where $\emptyset$ denotes the initial object in $\mathcal{S}$) and any object X in $\mathcal{S}$ the projections onto the second factor

$$\mathrm{pr}_2 : O(n) \times X \to X, \qquad n \geq 1,$$

together with the unique morphism $O(0) \to X$ make the discrete i-structure Δ_X over X into an i-O-algebra, which we denote by (Δ_X, δ_X). Indeed, the commutative diagram

$$\begin{tikzcd}
 & O(m)^n \times X^n \arrow[dr, "\cong"] & \\
O(m)^n \times X \arrow[ur, "1_{O(m)^n} \times \Delta_n"] \arrow[d, dashed, "\mathrm{pr}_2"'] & & (O(m) \times X)^n \arrow[d, "(\mathrm{pr}_2)^n"] \\
X \arrow[rr, tail, "\Delta_n"] & & X^n
\end{tikzcd}$$

shows that the neighbourhood axiom of definition 2.2.1 holds for $m \geq 1$. Since $O(0) \cong \emptyset$ (and since $\mathcal{S}$ is distributive) it also holds for $m = 0$. In the same vein, the associativity axiom holds, since the diagrams

$$\begin{tikzcd}
 & O(n) \times O(m)^n \times X \arrow[dl, "*_{nm} \times 1_X"'] \arrow[dr, dashed, "1_{O(n)} \times \mathrm{pr}_2"] & \\
O(m) \times X \arrow[dr, "\mathrm{pr}_2"'] & & O(n) \times X \arrow[dl, "\mathrm{pr}_2"] \\
 & X &
\end{tikzcd}$$

commute, and the projection axioms holds because of the commutative diagrams

$$\begin{tikzcd}
1 \times X \arrow[rr, "\pi_j^n \times 1_X"] \arrow[d, "\cong"'] & & O(n) \times X \arrow[d, "\mathrm{pr}_2"] \\
X \arrow[r, tail, "\Delta_n"] & X^n \arrow[r, "\mathrm{pr}_j"] & X
\end{tikzcd}$$

Clearly, any $\mathcal{S}$-morphism $h : X \longrightarrow Y$ becomes an i-O-homomorphism $(\Delta_X, \delta_X) \longrightarrow (\Delta_Y, \delta_Y)$. The discrete i-structure functor thus lifts to a functor $\Delta : \mathcal{S} \rightarrowtail O\text{-IAlg}(\mathcal{S})$, which is a section of the forgetful functor U. In particular, Δ is a fully faithful embedding of $\mathcal{S}$ into $O\text{-IAlg}(\mathcal{S})$.

In the case of the clone of affine combinations $\mathcal{A}_R$ the functor Δ is also a left adjoint of U. This is because the identity morphism 1_A induces a monomorphism $(\Delta_A, \delta_A) \rightarrowtail (A, \alpha)$ for any i-affine space (A, α); i.e. the subsequent diagrams commute

$$\begin{tikzcd}
\mathcal{A}_R(n) \times A \arrow[r, "1_{O(n)} \times \Delta_n"] \arrow[dr, "\mathrm{pr}_2"'] & \mathcal{A}_R(n) \times A\langle n \rangle \arrow[d, "\alpha_n"] \\
& A
\end{tikzcd}$$

where Δ_n denotes the factorisation of the diagonal map through $A\langle n \rangle \rightarrowtail A^n$. Since any i-affine map $h : (\Delta_X, \delta_X) \longrightarrow (A, \alpha)$ factors as $h : (\Delta_X, \delta_X) \xrightarrow{\Delta h} (\Delta_A, \delta_A) \xrightarrow{1_A} (A, \alpha)$ the natural bijections in proposition 2.1.2 thus lift to natural bijections

$$\mathcal{A}_R\text{-IAlg}(\mathcal{S})((\Delta_X, \delta_X), (A, \alpha)) \cong \mathcal{S}(X, A)$$

It remains to show the above diagram commutative. We shall show this by using the theory of i-affine space i$\mathbb{A}$. The commutativity of the diagram follows (in fact, is equivalent to) the i$\mathbb{A}$-provability of the sequent

$$\Lambda_n(\vec{\lambda}, P, \ldots, P, Q) \ \vdash_{\vec{\lambda}, P, Q} \ Q = P$$

We derive it from the associativity and projection axioms of $\mathrm{i}\mathbb{A}$: ($\vec{\delta}^n_j$ serves as the shorthand for $0,\ldots,0,\underset{j}{1},0,\ldots,0$)

$$
\begin{aligned}
\Lambda_n(\vec{\lambda},P,\ldots,P,Q) &\vdash_{\vec{\lambda},P,Q} (\exists!\vec{Y}) \bigwedge_{1\le j\le n} \Lambda_n(\vec{\delta}^n_j,P,\ldots,P,Y_j)\wedge\Lambda_n(\vec{\lambda},\vec{Y},Q)\\
&\vdash_{\vec{\lambda},P,Q} (\exists!\vec{Y}) \bigwedge_{1\le j\le n} (Y_j=P)\wedge\Lambda_n(\vec{\lambda},\vec{Y},Q)\\
&\vdash_{\vec{\lambda},P,Q} (\exists!\vec{Y}) \bigwedge_{1\le j\le n} \Lambda_1(1,P,Y_j)\wedge\Lambda_n(\vec{\lambda},\vec{Y},Q)\\
&\vdash_{\vec{\lambda},P,Q} \Lambda_1\Big(\sum_{j=1}^n \lambda_j\cdot 1,P,Q\Big)\\
&\vdash_{\vec{\lambda},P,Q} \Lambda_1(1,P,Q)\\
&\vdash_{\vec{\lambda},P,Q} Q=P
\end{aligned}
$$

Note that the projection axiom implies the $\mathrm{i}\mathbb{A}$-provable equivalence of

$$\Lambda_n(\vec{\delta}^n_j,P_1,\ldots,P_n,Q) \dashv\vdash_{\vec{P},Q} (Q=P_j)\wedge A\langle n\rangle(P_1,\ldots,P_n)$$

In particular, the formulae $(Y_j=P)$ and $\Lambda_1(1,P,Y_j)$ are $\mathrm{i}\mathbb{A}$-provably equivalent. We have used this in the third step.

The key property of i-affine spaces that makes this proof work is used in the fifth deduction step. The corresponding property of a clone O would be the equation $\sigma *_{n1} (\pi^1_1,\ldots,\pi^1_1)=\pi^1_1$. If a clone O satisfies this equation for each $n\ge 1$ and $O(0)\cong\emptyset$, then the same deduction can be made in the theory $\mathrm{i}\mathbb{C}$ to show Δ a left adjoint of U.

Finally, note that the discrete i-structure functor must have a further left adjoint since it preserves limits. This left adjoint must be a connected-components functor that, intuitively speaking, collapses infinitesimal neighbourhoods of each point to a point.

We have seen in proposition 1.3.9 that the total O-algebra $(O(0),*_{(-)0})$ is initial in O-$\mathrm{Alg}(\mathcal{S})$. The proof makes use of the associativity diagram and the fact that $A^0=1$ to show that for an O-algebra (A,α) the morphism α_0 is an O-algebra homomorphism. The same argument works for an i-O-algebra (A,α).

Proposition 2.6.9. *The total O-algebra $(O(0),*_{(-)0})$ is an initial object in O-$\mathrm{IAlg}(\mathcal{S})$.*

Proof. First of all, the commutative diagram of the neighbourhood axiom in definition 2.2.1 shows (for $m=0$) that $\alpha_0 : O(n)\to A\langle 1\rangle$ is an i-morphism. Since $A\langle 0\rangle\cong 1$ the associativity diagram for i-O-algebras then shows α_0 an i-O-homomorphism. The uniqueness of the i-O-homomorphism $(O(0), *_{(-)0}) \to (A, \alpha)$ follows as in the proof of proposition 1.3.9.

□

Remark 2.6.10 (*Initial i-affine space*)**.** For i-affine spaces over a non-trivial ring R the initial object is the empty affine space $\emptyset$. The forgetful functor $U : \mathrm{IAff}(\mathcal{S}) \to \mathcal{S}$ thus lifts (and hence preserves) initial objects. This is true for any clone O of a theory with no constants (i.e. where $O(0) \cong \emptyset$).

If R is a trivial ring then $U : i\mathbb{A}_R\text{-Mod}(\mathcal{S}) \to \mathcal{S}$ still lifts initial objects. However, the initial object of $\mathcal{A}_R\text{-IAlg}(\mathcal{S})$ is the total affine space of the equaliser $\{0 = 1\} \rightarrowtail R$ in $\mathcal{S}$ and different from $\emptyset$.

We study U-lifts of pushouts next. A pushout in Set can be constructed as the quotient of the coproduct $X \coprod Y$ by an equivalence relation, which is generated by the relation $\{(f(a), g(a)) \mid a \in A\}$.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "i_Y"] \\ X \arrow[r, "i_X"] & X \coprod_A Y \end{tikzcd}$$

The maps i_X and i_Y are the coproduct inclusions composed with the quotient map. Two elements $i_X(x)$ and $i_Y(y)$ are equal if and only if there are $a_1, \ldots, a_n$ in A and a *zig-zag*

$$\begin{tikzcd}[column sep=small] & a_1 \arrow[dl, "f"'] \arrow[dr, "g"] & & a_2 \arrow[dl, "g"'] \arrow[dr, "f"] & & \ldots & & a_n \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x = f(a_1) & & g(a_1) & & f(a_2) & \ldots & f(a_{n-1}) & & y = g(a_n) \end{tikzcd}$$

and similarly for $i_X(x)$ and $i_X(x')$ as well as $i_Y(y)$ and $i_Y(y')$. We formalise this description in *geometric logic* in the internal language of $\mathcal{S}$.

Lemma 2.6.11. *If the diagram in* $\mathcal{S}$

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "k"] \\ X \arrow[r, "h"] & Z \end{tikzcd}$$

is a pushout, then $\mathcal{S}$ satisfies the following sequents

$$\begin{aligned} \top &\vdash_{a:A} f(h(a)) = k(g(a)) \\ \top &\vdash_{z:Z} (\exists x : X)(h(x) = z) \vee (\exists y : Y)(k(y) = z) \\ h(x) = k(y) &\dashv\vdash_{x:X,y:Y} \bigvee_{j\in\mathbb{N}} zigzag_{2j+1} \\ h(x_0) = h(x_1) &\dashv\vdash_{x_0:X,x_1:X} \bigvee_{j\geq 1} zigzag_{2j} \\ k(y_0) = k(y_1) &\dashv\vdash_{y_0:Y,y_1:Y} \bigvee_{j\geq 1} \overline{zigzag}_{2j} \end{aligned}$$

where $zigzag_{2j+1}$ is a shorthand for the formula

$$(\exists a_1)\cdots(\exists a_{2j+1})\ x = f(a_1) \wedge g(a_1) = g(a_2) \wedge f(a_2) = f(a_3) \wedge \ldots$$
$$\ldots \wedge f(a_{2j}) = f(a_{2j+1}) \wedge g(a_{2j+1}) = y$$

$zigzag_{2j}$ is a shorthand for the formula

$$(\exists a_1)\cdots(\exists a_{2j})\ x_0 = f(a_1) \wedge g(a_1) = g(a_2) \wedge f(a_2) = f(a_3) \wedge \ldots$$
$$\ldots \wedge g(a_{2j-1}) = g(a_{2j}) \wedge f(a_{2j}) = x_1$$

and $\overline{zigzag}_{2j}$ is a shorthand for the formula

$$(\exists a_1)\cdots(\exists a_{2j})\ y_0 = g(a_1) \wedge f(a_1) = f(a_2) \wedge g(a_2) = g(a_3) \wedge \ldots$$
$$\ldots \wedge f(a_{2j-1}) = f(a_{2j}) \wedge g(a_{2j}) = y_1$$

Proof. The first sequent is stating that the square commutes. The second sequent is stating that h and k are jointly epimorphic. The other sequents can be obtained as follows. We know that Z is the quotient of the coproduct $X \coprod Y$ by the equivalence relation generated by the relation, which is the image of

$$A \xrightarrow{(i_X\circ f)\times(i_Y\circ g)} (X \coprod Y) \times (X \coprod Y)$$

This equivalence relation can be obtained by adding the diagonal, symmetrise and then form the transitive closure. (See, for example, [MM92, chap. 2 in appendix].) The zig-zag formulae describe the transitive closure after symmetrisation. □

Definition 2.6.12. Let (A,α) and (B,β) be i-O-algebras in $\mathcal{S}$. We say that an i-morphism $h : A \to B$ **reflects i-structure**, if $\mathcal{S}$ satisfies

$$B\langle n\rangle(h(x_1),\dots,h(x_n)) \vdash_{\vec{x}} A\langle n\rangle(x_1,\dots,x_n)$$

for all $n \in \mathbb{N}$. (The case $n = 0$ is trivial.)

Theorem 2.6.13. *The forgetful functor $U : O$-IAlg$(\mathcal{S}) \to \mathcal{S}$ lifts pushouts of spans of the form*

$$\begin{tikzcd} & (A,\alpha) \arrow[dl, "f"'] \arrow[dr, "g"] & \\ (C,\gamma) & & (B,\beta) \end{tikzcd}$$

uniquely, if f and g both reflect i-structure.

Proof. As before it is sufficient to show that U lifts such pushouts. Let

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "k"] \\ C \arrow[r, "h"] & Z \end{tikzcd}$$

be a pushout in $\mathcal{S}$. The i-O-algebra structure on Z will be constructed by taking the images of the i-O-algebra structures on C and B under h and k, respectively. We will use the characterisation of when the images of h and k coincide in terms of zig-zags together with the i-structure reflection properties of h and k to show that the so defined O-action is independent of the choice of representatives. Since h and k are jointly epimorphic the i-O-structure is well-defined.

For the sake of this approach we consider i-O-algebras as i$\mathbb{C}$-models over O and make use of the internal (geometric) language in $\mathcal{S}$. Even though the constructions and ideas are elementary their formalisation in geometric logic turns out to be rather tedious and lengthy. We will include the less formal and more intuitive proofs to help the reader to not get lost in the formal arguments.

(1) We define the i-structure $Z\langle n\rangle$, $n \in \mathbb{N}$ as the join of the images of $C\langle n\rangle$ and $B\langle n\rangle$ under h^n and k^n, respectively:

$$[\![\vec{z}.((\exists\vec{x})C\langle n\rangle(\vec{x}) \wedge \bigwedge_{1\le j\le n} h(x_j) = z_j) \vee ((\exists\vec{y})B\langle n\rangle(\vec{y}) \wedge \bigwedge_{1\le j\le n} k(y_j) = z_j)]\!] \rightarrowtail Z^n$$

For $n = 0$ this is the join of $C\langle 0\rangle$ and $B\langle 0\rangle$ as subobjects of 1; hence $Z\langle 0\rangle \cong 1$. The third sequent of lemma 2.6.11 shows $Z\langle 1\rangle = Z$. Let $r : m \to n$ be a map of finite, non-empty sets. Due to lemma 2.1.14(3) $\mathcal{S}$ satisfies the sequent

$$C\langle n\rangle(x_1,\dots,x_n) \wedge \bigwedge_{1\le j\le n} h(x_j) = z_j \ \vdash_{\vec{x},\vec{z}} \ C\langle m\rangle(x_{r(1)},\dots,x_{r(m)}) \wedge \bigwedge_{1\le j\le m} h(x_{r(j)}) = z_{r(j)}$$

and thus

$$(\exists\vec{x})C\langle n\rangle(x_1,\dots,x_n) \wedge \bigwedge_{1\le j\le n} h(x_j) = z_j \ \vdash_{\vec{z}} \ (\exists\vec{x})C\langle m\rangle(x_{r(1)},\dots,x_{r(m)}) \wedge \bigwedge_{1\le j\le m} h(x_{r(j)}) = z_{r(j)}$$

Applying the same reasoning to the i-structure B shows that $\mathcal{S}$ satisfies the sequent

$$(\exists\vec{y})B\langle n\rangle(y_1,\dots,y_n) \wedge \bigwedge_{1\le j\le n} k(y_j) = z_j \ \vdash_{\vec{z}} \ (\exists\vec{y})B\langle m\rangle(y_{r(1)},\dots,y_{r(m)}) \wedge \bigwedge_{1\le j\le m} k(y_{r(j)}) = z_{r(j)}$$

Taking the disjunction of the formulae on both sides of the turnstile finally shows

$$Z\langle n\rangle(z_1,\dots,z_n) \ \vdash_{\vec{z}} \ Z\langle m\rangle(z_{r(1)},\dots,z_{r(m)})$$

and we may conclude with lemma 2.1.14 that $Z\langle -\rangle$ is indeed an i-structure.

(2) We wish to define $\sigma \bullet^Z_n (z_1,\dots z_n)$ by $h(\sigma \bullet^C_n (x_1,\dots,x_n))$ or $k(\sigma \bullet^B_n (y_1,\dots,y_n))$, for x_j or y_j such that $z_j = h(x_j)$ or $z_j = k(y_j)$, respectively. For this purpose we consider (A,α), (B,β), (C,γ) as i$\mathbb{C}$-models over O, and denote the graphs of (the partial morphisms) α_n, β_n, γ_n by G^A_n, G^B_n and G^C_n, respectively. G^Z_n is defined as the interpretation of the formula

$$\begin{aligned} &((\exists x')(\exists\vec{x})G^C_n(\sigma,\vec{x},x') \wedge h(x') = w \wedge \bigwedge_{1\le j\le n} h(x_j) = z_j) \\ &\qquad \vee ((\exists y')(\exists\vec{y})G^B_n(\sigma,\vec{y},y') \wedge k(y') = w \wedge \bigwedge_{1\le j\le n} k(y_j) = z_j) \end{aligned}$$

with free variables $\sigma : O(n)$, $z_j : Z$, $w : Z$. The first functionality axiom

$$G^Z_n(\sigma,z_1,\dots,z_n,w) \ \vdash_{\sigma,\vec{z},w} \ Z\langle n\rangle(z_1,\dots,z_n)$$

follows from an argument similar to step (1). For example, the first functionality axiom for G_n^C yields the satisfaction of

$$(\exists x')(\exists \vec{x})G_n^C(\sigma, \vec{x}, x') \wedge h(x') = w \wedge \bigwedge_{1\leq j\leq n} h(x_j) = z_j \vdash_{\sigma,\vec{z},w} (\exists \vec{x})C\langle n\rangle(\vec{x}) \wedge \bigwedge_{1\leq j\leq n} h(x_j) = z_j$$

The respective sequent for $B\langle n\rangle$ is also valid due to the same reason. Taking the disjunction on both sides of the turnstiles of both sequents yields the first functionality axiom for G_n^Z.
In the same vein we obtain the existence part of the second functionality axiom; for applying the second functionality axiom to G_n^C validates the sequent

$$(\exists \vec{x})C\langle n\rangle(\vec{x}) \wedge \bigwedge_{1\leq j\leq n} h(x_j) = z_j \vdash_{\sigma,\vec{z}} (\exists \vec{x})(\exists x')G_n^C(\sigma, \vec{x}, x') \wedge \bigwedge_{1\leq j\leq n} h(x_j) = z_j$$

(Note that in geometric logic (or in any other fragment containing regular logic) existence is weaker than unique existence; i.e., $(\exists! x)\phi \vdash_x (\exists x)\phi$.) Since $\mathcal{S}$ satisfies the sequent $\top \vdash_{x':C} (\exists w)h(x') = w$, by Frobenius $\mathcal{S}$ also satisfies

$$\begin{aligned}(\exists \vec{x})(\exists x')G_n^C(\sigma, \vec{x}, x') \wedge \bigwedge_{1\leq j\leq n} h(x_j) = z_j \\ \vdash_{\sigma,\vec{z}} (\exists w)(\exists \vec{x})(\exists x')G_n^C(\sigma, \vec{x}, x') \wedge h(x') = w \wedge \bigwedge_{1\leq j\leq n} h(x_j) = z_j\end{aligned}$$

and thus

$$(\exists \vec{x})C\langle n\rangle(\vec{x}) \wedge \bigwedge_{1\leq j\leq n} h(x_j) = z_j \vdash_{\sigma,\vec{z}} (\exists w)(\exists \vec{x})(\exists x')G_n^C(\sigma, \vec{x}, x') \wedge h(x') = w \wedge \bigwedge_{1\leq j\leq n} h(x_j) = z_j$$

The respective sequent for G_n^B is satisfied as well, and as before we can take the disjunction to show the satisfaction of

$$Z\langle n\rangle(z_1, \ldots, z_n) \vdash_{\sigma,\vec{z}} (\exists w : Z)G_n^Z(\sigma, z_1, \ldots, z_n, w)$$

(Note that $(\exists x)(\phi \vee \psi)$ and $(\exists x)\phi \vee (\exists x)\psi$ are provably equivalent, so we may pull the existential quantifier in front. However, the same is not true for '$\exists!$'.) As for the uniqueness

$$G_n^Z(\sigma, \vec{z}, w) \wedge G_n^Z(\sigma, \vec{z}, w') \vdash_{\sigma,\vec{z},w,w'} w = w'$$

due to the distributivity law we obtain four cases of which two are symmetric. The resulting two cases amount to show that $\mathcal{S}$ satisfies

$$(\exists \vec{x}_0)(\exists \vec{x}_1)(\exists x'_0)(\exists x'_1) G^C_n(\sigma, \vec{x}_0, x'_0) \wedge h(x'_0) = w \wedge \bigwedge_{1 \le j \le n} h(x_{0,j}) = z_j$$
$$\wedge\, G^C_n(\sigma, \vec{x}_1, x'_1) \wedge h(x'_1) = w' \wedge \bigwedge_{1 \le j \le n} h(x_{1,j}) = z_j \ \vdash_{\sigma, \vec{z}, w, w'} \ w = w'$$

$$(\exists \vec{x})(\exists \vec{y})(\exists x')(\exists y') G^C_n(\sigma, \vec{x}, x') \wedge h(x') = w \wedge \bigwedge_{1 \le j \le n} h(x_j) = z_j$$
$$\wedge\, G^B_n(\sigma, \vec{y}, y') \wedge k(y') = w' \wedge \bigwedge_{1 \le j \le n} h(y_j) = z_j \ \vdash_{\sigma, \vec{z}, w, w'} \ w = w'$$

The proofs of both sequents are similar, hence we shall only give a proof of the second sequent. Due to the transitivity of equality we can apply the third sequent of lemma 2.6.11 to each $h(x_{1,j}) = h(y_j)$, $1 \le j \le n$. Note that $zigzag_{2\ell+1} \vdash_{x,y} zigzag_{2\ell'+1}$ for every $\ell \le \ell'$, since any zig-zag can be trivially extended. Hence, after applying the distributivity law we can simplify to

$$(\exists \vec{x})(\exists \vec{y})(\exists x')(\exists y')\ G^C_n(\sigma, \vec{x}, x') \wedge h(x') = w \wedge \bigwedge_{1 \le j \le n} h(x_j) = z_j$$
$$\wedge\, G^B_n(\sigma, \vec{y}, y') \wedge k(y') = w' \wedge \bigwedge_{1 \le j \le n} h(y_j) = z_j$$
$$\vdash_{\sigma, \vec{z}, w, w'}$$
$$\bigvee_{\ell \ge 0} (\exists \vec{x})(\exists \vec{y})(\exists x')(\exists y')\ G^C_n(\sigma, \vec{x}, x') \wedge h(x') = w$$
$$\wedge\, G^B_n(\sigma, \vec{y}, y') \wedge k(y') = w' \wedge \bigwedge_{1 \le j \le n} zigzag_{2\ell+1}[x_j, y_j / x, y]$$

It suffices to show that $\mathcal{S}$ satisfies

$$\bigvee_{\ell \ge 0} (\exists \vec{a}_1) \cdots (\exists \vec{a}_{2\ell+1})(\exists \vec{x})(\exists \vec{y})(\exists x')(\exists y')\ G^C_n(\sigma, \vec{x}, x') \wedge h(x') = w \wedge G^B_n(\sigma, \vec{y}, y') \wedge k(y') = w'$$
$$\wedge \bigwedge_{1 \le j \le n} zigzag'_{2\ell+1}[x_j, y_j / x, y] \ \vdash_{\sigma, \vec{z}, w, w'} \ w = w'$$

where $zigzag'_{2\ell+1}$ denote the zig-zag formula with the existential quantifiers removed. Before we proceed with the formal proof, we give a naive proof first. Note that from the zig-zags we find $a^j_1 : A$ such that $x_j = f(a^j_1)$. But $B\langle n \rangle(x_1, \ldots, x_n)$ holds true, so, since f reflects i-structure,

$A\langle n\rangle(a_1^1, \ldots, a_1^n)$ holds true as well. The morphisms g and f both preserve and reflect i-structure, so an easy induction over ℓ shows that $A\langle n\rangle(a_i^1, \ldots, a_i^n)$ holds true for each $1 \leq i \leq 2\ell + 1$. Since f and g are i-O-homomorphisms this yields a zig-zag

$$\begin{tikzcd}[column sep=tiny]
& \sigma \bullet_n^A (\vec{a}_1) \arrow[dl, "f"'] \arrow[dr, "g"] & & \cdots & & \sigma \bullet_n^A (\vec{a}_{2\ell+1}) \arrow[dl, "f"'] \arrow[dr, "g"] & \\
\sigma \bullet_n^B (\vec{x}) & & \sigma \bullet_n^C g^n(\vec{a}_1) & \cdots & \sigma \bullet_n^B f^n(\vec{a}_{2\ell}) & & \sigma \bullet_n^C (\vec{y})
\end{tikzcd}$$

From lemma 2.6.11 we conclude that

$$h(\sigma \bullet_n^B (x_1, \ldots, x_n)) = k(\sigma \bullet_n^C (y_1, \ldots, y_n))$$

as required. In terms of sequents the fact that f and g are i-O-homomorphisms that reflect i-structure translates into the $\mathcal{S}$-provable equivalence

$$(\exists \vec{x})\ G_n^C(\sigma, \vec{x}, x') \wedge \bigwedge_{1\leq j\leq n} x_j = f(a^j) \ \dashv\vdash_{\sigma,\vec{a},x'} \ (\exists a')\ G_n^A(\sigma, \vec{a}, a') \wedge x' = f(a')$$

and similarly for G_n^B. Applying this to each $\vec{a}_i$, $1 \leq i \leq 2\ell + 1$ in the conjunction over the zig-zags we construct a zig-zag formula $zigzag_{2\ell+1}[x', y'/x, y]$ and hence make the following deduction

$$\begin{aligned}
(\exists \vec{a}_1) \cdots (\exists \vec{a}_{2\ell+1})(\exists \vec{x})(\exists \vec{y})(\exists x')(\exists y')\ & G_n^C(\sigma, \vec{x}, x') \wedge h(x') = w \wedge G_n^B(\sigma, \vec{y}, y') \wedge k(y') = w' \\
& \wedge \bigwedge_{1\leq j\leq n} zigzag'_{2\ell+1}[x_j, y_j/x, y] \\
& \vdash_{\sigma,\vec{z},w,w'} \ (\exists x')(\exists y')\ h(x') = w \wedge k(y') = w' \wedge zigzag_{2\ell+1}[x', y'/x, y] \\
& \vdash_{\sigma,\vec{z},w,w'} \ (\exists x')(\exists y')\ h(x') = w \wedge k(y') = w' \wedge h(x') = k(y') \\
& \vdash_{\sigma,\vec{z},w,w'} \ w' = w
\end{aligned}$$

(3) We discuss the remaining axioms of an i$\mathbb{C}$-model over O. The neighbourhood axiom holds because we can find (uniform) representatives ($x_j : C$, say), such that $h(x_i) = z_i$ and

$$\sigma_j \bullet_n^Z (z_1, \ldots, z_n) = h(\sigma_j \bullet_n^C (x_1, \ldots, x_n))$$

for all $1 \leq j \leq n$. In fact, the neighbourhood axiom follows from the neighbourhood axioms for C and B and an argument like in step (2). More formally and in terms of sequents we first apply the distributivity law to $\bigwedge_{1\le j\le n} G^Z_m(\sigma_j, \vec{z}, w_j)$. The resulting disjunction will have a conjunction of the formulae with G^C_m only, a conjunction of the formulae with G^B_m only and conjunctions of mixed factors. We consider the mixed factors first, say

$$\bigwedge_{1\le j\le n_0} (\exists x'_j)(\exists \vec{x}^{\,j})\ G^C_m(\sigma_j, \vec{x}^{\,j}, x'_j) \wedge h(x'_j) = w_j \wedge \bigwedge_{1\le \ell\le m} h(x^j_\ell) = z_\ell$$
$$\wedge \bigwedge_{n_0<i\le n} (\exists y'_i)(\exists \vec{y}^{\,i})\ G^B_m(\sigma_i, \vec{y}^{\,i}, y'_i) \wedge k(y'_i) = w_j \wedge \bigwedge_{1\le \ell\le m} k(y^i_\ell) = z_\ell$$

where $1 \le n_0 < n$. Let us write ϕ for this formula. For each pair (j, i), $1 \le j \le n_0 < i \le n$ we have

$$\phi \vdash_{\vec{\sigma},\vec{z},\vec{w}} (\exists \vec{x}^{\,j})(\exists \vec{y}^{\,i}) \bigwedge_{1\le \ell\le m} h(x^j_\ell) = k(y^i_\ell)$$

As in step (2) we can apply lemma 2.6.11. The rest of the proof is similar to step (2): using the zig-zags and that f and g reflect and preserve i-structure we find that

$$(\exists x')(\exists \vec{x}^{\,j})\ G^C_m(\sigma_i, \vec{x}^{\,j}, x') \wedge h(x') = w \wedge \bigwedge_{1\le \ell\le m} h(x^j_\ell) = z_\ell$$

is $\mathcal{S}$-provably equivalent to

$$(\exists y')(\exists \vec{y}^{\,i})\ G^B_m(\sigma_i, \vec{y}^{\,i}, y') \wedge k(y') = w \wedge \bigwedge_{1\le \ell\le m} h(y^i_\ell) = z_\ell$$

In fact, we can apply the same argument also to $\vec{x}^{\,j}$ and $\vec{x}^{\,j'}$, and hence find one uniform representative $\vec{x}$ for all σ_j. In other words, ϕ is $\mathcal{S}$-provably equivalent to

$$\bigwedge_{1\le j\le n} (\exists x_j)(\exists \vec{x})\ G^C_m(\sigma_j, \vec{x}, x_j) \wedge h(x_j) = w_j \wedge \bigwedge_{1\le \ell\le m} h(x^j_\ell) = z_\ell$$

Similarly, the case of the conjunction of formulae with G^C_m only is $\mathcal{S}$-provably equivalent to this formula. The remaining case is $\mathcal{S}$-provably equivalent to this formula as well, if we replace $G^C_m(\sigma_j, \vec{x}, x_j)$ by $G^B_m(\sigma_j, \vec{x}, x_j)$. The neighbourhood axiom now follows from the fact that it holds for G^C_m and G^B_m.

The associativity axiom is obtained by the same kind of reasoning. We find representatives ($x_j : C$, say), such that $h(x_i) = z_i$, then

$$
\begin{aligned}
(\sigma *_{nm} (t_1, \dots, t_n)) \bullet_n^Z (z_1, \dots, z_n) &= h((\sigma *_{nm} (t_1, \dots, t_n)) \bullet_n^C (x_1, \dots, x_n)) \\
&= h(\sigma \bullet_n^C (t_1 \bullet_m^C x_1, \dots, t_n \bullet_m^C x_n)) \\
&= \sigma \bullet_n^Z (h(t_1 \bullet_m^C x_1), \dots, h(t_n \bullet_m^C x_n)) \\
&= \sigma \bullet_n^Z (t_1 \bullet_m^Z z_1, \dots, t_n \bullet_m^Z z_n)
\end{aligned}
$$

by the associativity axiom for C (or B), respectively. More formally and with sequents, it is sufficient to show

$$G_m^Z(\sigma *_{nm} (t_1, \dots, t_n), \vec{z}, w) \ \vdash_{\sigma,\vec{t},\vec{z},w} \ (\exists \vec{z}') \bigwedge_{1 \le j \le n} G_m^Z(t_j, \vec{z}', w_j') \wedge G_n^Z(\sigma, \vec{z}', w)$$

(Since both formulae are provably functional with the same domain and codomain, the provable equivalence follows from this already.) However, this sequent can be easily deduced from the associativity axioms for G_n^C and G_n^B. The projection axiom is also a direct consequence of the projection axioms for G_n^C and G_n^B.

(4) Clearly, the morphisms k and h become i-O-homomorphisms by construction of the i-O-structure on Z. It remains to show that given an i-O-algebra (W, γ) and i-O-homomorphisms r and s rendering the subsequent diagram commutative

$$
\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "k"] \arrow[ddr, bend left, "s"] & \\
C \arrow[r, "h"] \arrow[drr, bend right, "r"'] & Z \arrow[dr, dashed, "t"] & \\
& & W
\end{tikzcd}
$$

the unique dashed $\mathcal{S}$-morphism t becomes an i-O-homomorphism. In the naive approach this is clear from the construction of the i-O-structure on Z. For the formal proof we shall consider W as an i$\mathbb{C}$-model over O. The graphs of γ_n shall be denoted by G_n^W. The morphism t is an i-morphism, if the following sequent is satisfied for every $n \ge 1$

$$((\exists \vec{x})\ C\langle n\rangle(\vec{x}) \wedge \bigwedge_{1 \le j \le n} h(x_j) = z_j) \vee ((\exists \vec{y})\ B\langle n\rangle(\vec{y}) \wedge \bigwedge_{1 \le j \le n} k(y_j) = z_j) \ \vdash_{\vec{z}} \ W\langle n\rangle(t^n(\vec{z}))$$

A deduction of the first case uses that r is is an i-morphism

$$
\begin{aligned}
(\exists \vec{x})\ C\langle n\rangle(\vec{x}) \wedge \bigwedge_{1\leq j\leq n} h(x_j) = z_j \vdash_{\vec{z}}\ & (\exists \vec{x})\ C\langle n\rangle(\vec{x}) \wedge \bigwedge_{1\leq j\leq n} th(x_j) = t(z_j) \\
\vdash_{\vec{z}}\ & (\exists \vec{x})\ C\langle n\rangle(\vec{x}) \wedge \bigwedge_{1\leq j\leq n} r(x_j) = t(z_j) \\
\vdash_{\vec{z}}\ & (\exists \vec{x})\ W\langle n\rangle(r^n(\vec{x})) \wedge \bigwedge_{1\leq j\leq n} r(x_j) = t(z_j) \\
\vdash_{\vec{z}}\ & (\exists \vec{x})\ W\langle n\rangle(t(z_1), \dots, t(z_n)) \\
\vdash_{\vec{z}}\ & W\langle n\rangle(t(z_1), \dots, t(z_n))
\end{aligned}
$$

The second case is obtained in the same way. To show t an i-O-homomorphism we need to proof the sequent

$$(\exists z)\ G_n^Z(\sigma, z_1, \dots, z_n, z) \wedge t(z) = w \vdash_{\sigma,\vec{z},w} G_n^W(\sigma, t(z_1), \dots, t(z_n), w)$$

We will only give a proof of the first case.

$$
\begin{aligned}
&(\exists z)(\exists x)(\exists \vec{x})\ G_n^C(\sigma, \vec{x}, x) \wedge h(x) = z \wedge t(z) = w \wedge \bigwedge_{1\leq j\leq n} h(x_j) = z_j \\
&\qquad \vdash_{\sigma,\vec{z},w}\ (\exists x)(\exists \vec{x})\ G_n^C(\sigma, \vec{x}, x) \wedge r(x) = w \wedge \bigwedge_{1\leq j\leq n} h(x_j) = z_j \\
&\qquad \vdash_{\sigma,\vec{z},w}\ (\exists \vec{x})\ G_n^W(\sigma, r^n(\vec{x}), w) \wedge \bigwedge_{1\leq j\leq n} h(x_j) = z_j \\
&\qquad \vdash_{\sigma,\vec{z},w}\ (\exists \vec{x})\ G_n^W(\sigma, r^n(\vec{x}), w) \wedge \bigwedge_{1\leq j\leq n} r(x_j) = t(z_j) \\
&\qquad \vdash_{\sigma,\vec{z},w}\ G_n^W(\sigma, t(z_1), \dots, t(z_n), w)
\end{aligned}
$$

□

Corollary 2.6.14. *U lifts (small) wide pushouts of i-structure reflecting i-O-homomorphisms uniquely.*

Proof. (Small) wide pushouts are filtered colimits of finite wide pushouts, and finite wide pushouts can be constructed by iterating ordinary pushouts. We need to show that a pushout of i-structure reflecting i-O-homomorphism is i-structure reflecting. The assertion then follows from combining theorem 2.6.13 and proposition 2.6.6.

We want to show that h and k in the pushout diagram of theorem 2.6.13 reflect i-structure. It suffices to show that this is true for h. We shall give the naive proof only. It is the standard argument using zig-zags we have applied several times already. (A more formal argument follows a similar pattern as the proof of unique existence in step (2) of the proof of theorem 2.6.13.) Suppose $Z\langle n\rangle(h(x_1), \dots, h(x_n))$ holds, then, without loss of generality, there are $x'_j : C$ such that $h(x_j) = h(x'_j)$ and $C\langle n\rangle(x'_1, \dots, x'_n)$. By lemma 2.6.11 we find zig-zags

$$\begin{tikzcd}
& a_1^j \arrow[dl, "f"'] \arrow[dr, "g"] & & a_2^j \arrow[dl, "g"'] \arrow[dr, "f"] & & \cdots & & a_n^j \arrow[dl, "g"'] \arrow[dr, "f"] & \\
x'_j & & g(a_1^j) & & f(a_2^j) & \cdots & g(a_{n-1}^j) & & x_j
\end{tikzcd}$$

for each $1 \leq j \leq n$, and we may assume that they are all of the same length n. Since f and g reflect and preserve i-structure we can see by following the zig-zag paths that $C\langle n\rangle(x'_1, \dots, x'_n)$ implies $C\langle n\rangle(x_1, \dots, x_n)$. □

Lemma 2.6.15. *Let (A, α) be a total O-algebra. Any i-morphism $h : A \longrightarrow B$ reflects i-structure.*

Proof. $\mathcal{S}$ satisfies the sequent $B\langle n\rangle(h(x_1), \dots, h(x_n)) \vdash_{\vec{x}} \top$ due to the soundness of the conjunction rules. This sequent states that h reflects i-structure, for A is a total i-structure. □

Corollary 2.6.16. *U lifts (small) wide pushouts uniquely, if the vertex of the span is a total O-algebra.*

Remark 2.6.17 (*Coproducts*)**.** (Small) coproducts are pushouts of (wide) spans with the vertex being an initial object. Since the initial object in O-IAlg$(\mathcal{S})$ is a total O-algebra we can apply corollary 2.6.16 to describe coproducts in O-IAlg$(\mathcal{S})$ in terms of pushouts in $\mathcal{S}$. In particular, U lifts (and hence preserves) small coproducts if and only if $O(0) \cong \emptyset$. For a non-trivial ring R, $U : \text{IAff}(\mathcal{S}) \longrightarrow \mathcal{S}$ thus lifts small coproducts.

Lemma 2.6.18. *Let $O(0) \cong \emptyset$, then U lifts coequalisers of pairs of i-structure reflecting i-O-homomorphisms uniquely.*

Proof.

$$\begin{tikzcd}
U(A, \alpha) \arrow[r, shift left, "Uh"] \arrow[r, shift right, "Ug"'] & U(B, \beta) \arrow[r, "q"] & Z
\end{tikzcd}$$

Let h and g reflect i-structure, and let q be a coequaliser of h and g in $\mathcal{S}$. We can write the coequaliser equivalently as a pushout

$$\begin{tikzcd}
A \coprod B \arrow[r, "\{h,1_B\}"] \arrow[d, "\{g,1_B\}"'] & B \arrow[d, "q"] \\
B \arrow[r, "q"] & Z
\end{tikzcd}$$

Since $O(0) \cong \emptyset$ the forgetful functor U lifts coproducts. Moreover, since h, g and 1_B reflect i-structure, $\{g, 1_B\}$ and $\{h, 1_B\}$ reflect i-structure as well. By theorem 2.6.13 this pushout diagram has a unique lift. Since the U-lift of the coproduct on the top left is a coproduct in O-IAlg($\mathcal{S}$), the lifted pushout is also a lift of the coequaliser q.

□

Theorem 2.6.19. *Let O be a clone in $\mathcal{S}$ with $O(0) \cong \emptyset$ and I a small category. U lifts colimits of diagrams $D : I \to O$-IAlg($\mathcal{S}$) of i-structure reflecting i-O-homomorphisms uniquely. Moreover, the morphisms of the colimit cone all reflect i-structure.*

Proof. A colimit Z with limiting cocone λ of $U \circ D$ is a coequaliser q of the morphisms P, Q (cf. the dual of [Mac98, thm. V.2.1])

$$\begin{tikzcd}
\operatorname{dom} UD(h) \arrow[r, "1_{\operatorname{dom}(UD(h))}"] \arrow[d, "i_h"'] & \operatorname{dom}(UD(h)) \arrow[d, "i_{\operatorname{dom}(UD(h))}"] \arrow[dr, bend left, "\lambda_{\operatorname{dom}(D(h))}"] & \\
\coprod_{k \in \operatorname{mor}(I)} \operatorname{dom}(UD(k)) \arrow[r, shift left, "P"] \arrow[r, shift right, "Q"'] & \coprod_{j \in \operatorname{ob}(I)} UD(i) \arrow[r, two heads, "q"] & Z \\
\operatorname{dom}(UD(k)) \arrow[u, "i_k"] \arrow[r, "UD(k)"'] & \operatorname{cod}(UD(k)) \arrow[u, "i_{\operatorname{cod} UD(f)}"'] \arrow[ur, bend right, "\lambda_{\operatorname{cod}(D(k))}"'] &
\end{tikzcd}$$

Since $O(0) \cong \emptyset$ the forgetful functor lifts small coproducts. Moreover, Since the coproduct inclusions reflect i-structure (see the proof of corollary 2.6.14), the i-O-homomorphism P reflects i-structure. Since the i-O-homomorphisms $D(k)$ reflect i-structure, the i-O homomorphism Q reflects it as well. By the preceding lemma U lifts the coequaliser q uniquely. Since it also lifts the coproducts the λ_j have U-lifts, too. This yields a lift of the cocone λ and shows the U-lift of Z and λ a colimit of D.

Finally, the proof of the preceding lemma together with the proof of corollary 2.6.14 show that Q reflects i-structure. Since the coproduct inclusions reflect i-structure, each $\lambda_j : D(j) \to Z$ has to reflect i-structure as well. □

We characterise U-lifts of quotients (i.e. coequalisers) of equivalence relations next. Recall that an *equivalence relation* in a finite-limit category is a model of the (Horn) theory with one sort A, one binary relation $R \rightarrowtail AA$ and axioms

$$\begin{aligned} \top &\vdash_x R(x,x) \\ R(x,y) &\vdash_{x,y} R(y,x) \\ R(x,y) \wedge R(y,z) &\vdash_{x,y,z} R(x,z) \end{aligned}$$

Theorem 2.6.20. *Let* $(R,\rho) \overset{p_1}{\underset{p_2}{\rightrightarrows}} (A,\alpha)$ *be an equivalence relation in* O-$\mathrm{IAlg}(\mathcal{S})$*.* U *lifts the quotient of* R *uniquely if and only if* p_1 *and* p_2 *jointly reflect the i-structure, that is* $\mathcal{S}$ *satisfies*

$$A\langle n\rangle(p_1(x_1),\ldots,p_1(x_n)) \wedge A\langle n\rangle(p_2(x_1),\ldots,p_2(x_n)) \vdash_{\vec{x}} R\langle n\rangle(x_1,\ldots,x_n), \qquad n \in \mathbb{N}$$

Proof. (1)

$$R \overset{p_1}{\underset{p_2}{\rightrightarrows}} A \xrightarrow{q} Z$$

Since U preserves finite limits, R is an equivalence relation in $\mathcal{S}$. Let q be its quotient. Firstly, we show that p_1 and p_2 jointly reflecting i-structure is sufficient for the existence of a U-lift of the coequaliser, which is then necessarily unique. As in the proof of theorem 2.6.20 the i-O-structure on Z is constructed by taking the respective images of the i-O-structure under q. The i-structure $Z\langle n\rangle$ is thus defined as

$$[\![\vec{z}.(\exists\vec{x})\ A\langle n\rangle(x_1,\ldots,x_n) \wedge \bigwedge_{1\le j\le n} q(x_i) = z_i]\!] \rightarrowtail Z^n$$

and the O-action γ is defined by the graphs G_n^Z, which are the interpretations

$$[\![\sigma,\vec{z},z.(\exists x)(\exists\vec{x})\ G_n^A(\sigma,x_1,\ldots,x_n,x) \wedge q(x) = z \wedge \bigwedge_{1\le j\le n} q(x_i) = z_i]\!] \rightarrowtail O(n)\times Z^n \times Z$$

We shall only give the naive proof that this makes Z into an i$\mathbb{C}$-model over O and hence an i-O-algebra. For this purpose we will denote the respective O-actions by '$\bullet_n^R$', '$\bullet_n^A$' and '$\bullet_n^Z$'.

(2) Since $A\langle -\rangle$ is an i-structure and q an epimorphism, it is clear that $Z\langle -\rangle$ defines an i-structure for which q is an i-morphism. We need to show that $\sigma \bullet_n^Z (z_1,\ldots,z_n)$ is well-defined for $\sigma : O(n)$ and $z_i : Z$. By the definition of $Z\langle n\rangle$ there are $x_i : A$ such that $q(x_i) = z_i$,

$A\langle n\rangle(x_1, \ldots, x_n)$ holds and (by definition)

$$\sigma \bullet_n^Z (z_1, \ldots, z_n) = q(\sigma \bullet_n^A (x_1, \ldots, x_n))$$

Let $x'_i : A$ be such that $q(x'_i) = q(x_i)$ and $A\langle n\rangle(x'_1, \ldots, x'_n)$ holds. Since $\mathcal{S}$ is an effective regular category, R is the kernel pair of q, and hence $R(x_i, x'_i)$ holds. This means that there are (uniquely determined) $y_i : R$ such that $p_1(y_i) = x_i$ and $p_2(y_i) = x'_i$. Moreover, $A\langle n\rangle(x_1, \ldots, x_n)$ and $A\langle n\rangle(x'_1, \ldots, x'_n)$ both hold true and thus does $R\langle n\rangle(x_1, \ldots, x_n)$; for p_1 and p_2 jointly reflect i-structure. Since p_1 and p_2 are i-O-homomorphisms we have

$$\begin{aligned}\sigma \bullet_n^A (x_1, \ldots, x_n) &= p_1(\sigma \bullet_n^R (y_1, \ldots, y_n))\\ \sigma \bullet_n^A (x'_1, \ldots, x'_n) &= p_2(\sigma \bullet_n^R (y_1, \ldots, y_n))\end{aligned}$$

and thus $q(\sigma \bullet_n^A (x_1, \ldots, x_n)) = q(\sigma \bullet_n^A (x'_1, \ldots, x'_n))$.

(3) The axioms of an i-O-algebra for $(Z, \bullet^Z)$ follow easily from the fact that they hold for $(A, \bullet^A)$. For example, for the neighbourhood axiom we need to show

$$Z\langle n\rangle(\sigma_1 \bullet_m^Z (z_1, \ldots, z_m), \ldots, \sigma_n \bullet_m^Z (z_1, \ldots, z_m))$$

for $n, m \in \mathbb{N}$. Let $x_j : A$ be such that $A\langle m\rangle(x_1, \ldots, x_m)$ holds and $q(x_i) = z_i$, then the said relation holds, for we have

$$A\langle n\rangle(\sigma_1 \bullet_m^A (x_1, \ldots, x_m), \ldots, \sigma_n \bullet_m^A (x_1, \ldots, x_m))$$

The associativity and projection axioms are obtained in the same way.

(4) Clearly, q lifts to an i-O-homomorphism $(A, \bullet^A) \longrightarrow (Z, \bullet^Z)$ and the universal property of q as the coequaliser of p_1 and p_2 in O-IAlg$(\mathcal{S})$ follows easily as well. It remains to show that the condition of p_1 and p_2 jointly reflecting i-structure is necessary. Assume that the coequaliser in $\mathcal{S}$

$$U(R, \rho) \underset{Up_2}{\overset{Up_1}{\rightrightarrows}} U(A, \alpha) \xrightarrow{\;q\;} Z$$

has a U-lift. We know that $U(R, \rho)$ is also the kernel pair of q. Since U lifts kernel pairs uniquely by theorem 2.6.4, (R, ρ) is a kernel pair of q in O-IAlg$(\mathcal{S})$. It has the property that the commutative square

$$\begin{tikzcd} R\langle n\rangle \arrow[r, tail, "(p_1^n, p_2^n)"] \arrow[d, tail] & A\langle n\rangle \times A\langle n\rangle \arrow[d, tail] \\ R^n \arrow[r, tail, "(p_1^n, p_2^n)"] & A^n \times A^n \end{tikzcd} \tag{2.6}$$

is a pullback (in $\mathcal{S}$) for $n \in \mathbb{N}$. This follows from a diagram chase and the fact that the two top and bottom diagrams in

$$\begin{tikzcd} R\langle n\rangle \arrow[r, shift left, "p_1^n"] \arrow[r, shift right, "p_2^n"'] \arrow[d, tail] & A\langle n\rangle \arrow[r, "q^n"] \arrow[d, tail] & Z\langle n\rangle \arrow[d, tail] \\ R^n \arrow[r, shift left, "p_1^n"] \arrow[r, shift right, "p_2^n"'] & A^n \arrow[r, "q^n"] & Z^n \end{tikzcd}$$

are kernel pairs. The diagram (2.6) being a pullback is the diagrammatic way of saying that $p_1$and p_2 jointly reflect the i-structure, which can be seen easily from the interpretation of the respective sequent in $\mathcal{S}$.

□

Corollary 2.6.21. *O-IAlg($\mathcal{S}$) is a regular category and U is a regular functor that reflects regular epimorphisms.*

Proof. A regular epimorphism is the coequaliser of its kernel pair. Since both $\mathcal{S}$ and O-IAlg($\mathcal{S}$) have kernel pairs, the preceding theorem shows that an i-O-homomorphism h is a regular epimorphism in O-IAlg($\mathcal{S}$), if and only if Uh is a regular epimorphism in $\mathcal{S}$. In other words, U preserves and reflects regular epimorphisms. Since it preserves finite limits U is a regular functor. Regular epimorphisms are stable under pullback in $\mathcal{S}$. Since U preserves pullbacks and preserves and reflects regular epimorphisms, regular epimorphisms are stable under pullback in O-IAlg($\mathcal{S}$) as well. Since O-IAlg($\mathcal{S}$) also has all finite limits and coequalisers it is, in particular, a regular category. □

Remarks 2.6.22.

(a) Not every epimorphism in O-IAlg($\mathcal{S}$) needs to be regular. Consider, for example, an i-affine space (A, α) over a non-trivial ring R, which is not isomorphic to a discrete i-affine space. The epimorphism $1_A : (\Delta_A, \delta_X) \to (A, \alpha)$ is not regular, since it is a monomorphism, but not an isomorphism.

(b) Discrete i-affine spaces also show that not every equivalence relation in IAff($\mathcal{S}$) is the kernel pair of some i-affine morphism. Indeed, let (A, α) be a total affine space such that A

is well-supported in $\mathcal{S}$ (i.e. the terminal map is an epimorphism) and A is not a terminal object. The subobject $1_{A\times A} : (\Delta_{A\times A}, \delta_{A\times A}) \rightarrowtail (A,\alpha)\times(A,\alpha)$ is an equivalence relation, which is not the kernel pair of an i-affine map.

Indeed, assume there were an i-affine morphism $f : (A,\alpha) \to (B,\beta)$ such that

$$(A\times A, \delta_{A\times A}) \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} (A,\alpha)$$

is the kernel pair of f. Since U preserves kernel pairs and the diagram

$$A\times A \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} A \longrightarrow 1$$

is *exact* (i.e. the two projections are a kernel pair of the terminal map, and the terminal map is a quotient in $\mathcal{S}$, since A is well-supported), the morphism f factors through 1. Hence $(A\times A, \delta_{A\times A}) \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} (A,\alpha)$ is the kernel pair of the terminal map $(A,\alpha)\to 1$, and thus $1_{A\times A} : (A\times A, \delta_{A\times A}) \to (A,\alpha)\times(A,\alpha)$ must be an isomorphism. But then the diagonal map $\Delta : A \to A\times A$ must be an isomorphism as well and A is subterminal, which yields a contradiction to our choice of A.

This shows that the category $\mathrm{IAff}(\mathcal{S})$ is not effective regular.

Chapter 3

Applications

In this chapter we shall give some applications of infinitesimal algebra to *Synthetic Differential Geometry* (SDG).

We begin with the study of C^∞*-rings* and show that parallel to the theory of K-algebras every Horn formula-in-context in the syntactic category of C^∞-rings is an i-affine space over its nil-square i-structure, and that every morphism between Horn formulae-in-context is an i-affine map. As for K-algebras, this result extends to the opposite category of C^∞-rings. In particular, we obtain that every smooth paracompact manifold admits an i-affine structure, and that every smooth map is i-affine, when we embed manifolds into C^∞-rings.

We then turn to *well-adapted models* of SDG. Using the classifying topos of archimedean local C^∞-rings we show that each well-adapted model has a good supply of i-affine spaces. In particular, every smooth manifold carries an i-affine structure over a nil-square i-structure when embedded into the topos of the well-adapted model, and every smooth map embeds into an i-affine map.

Formal manifolds form an important class of geometric objects in SDG. (Cf. [Koc09]) We show that formal manifolds are i-affine spaces in a natural way: the i-affine structure is glued together from the i-affine structures of their charts, and the i-affine structure of the charts is inherited from the i-affine nil-square structure on R^n. Moreover, any morphism between formal manifolds is i-affine. A parallel result should be true for schemes in algebraic geometry, and we give an outline of the proof. Besides formal manifolds, another important class of geometric objects are *infinitesimally smooth spaces* (also known as microlinear spaces). With the infinitesimal cross we shall give a non-trivial example of an i-affine space that is not infinitesimally smooth.

We conclude this chapter and this thesis by developing some basic geometric i-algebra of loci and formal 1-manifolds in the context of naive SDG. Firstly, we introduce formal 1-manifolds and loci, show that they are i-affine spaces, and prove that all maps between them are i-affine. Secondly, we study the tangent bundle over an i-affine space and develop the R-linear structure on each tangent space from the i-affine structure of the space step by step. With this we hope to give a glance how infinitesimal geometric algebra can be used to study and develop structures and concepts in SDG.

3.1 C$^\infty$-Rings

The theory of C$^\infty$*-rings* is the one-sorted algebraic theory with the n-ary operations being smooth real functions $f : \mathbb{R}^n \to \mathbb{R}$. It is most naturally defined as a Lawvere theory L with objects $\mathbb{R}^n$ for each $n \in \mathbb{N}$ and morphisms smooth functions $f : \mathbb{R}^n \to \mathbb{R}^m$ [MR91, chap. I.1]. However, since we want to work synthetically and with syntactic categories, we shall define C$^\infty$-rings using the syntactic approach.

Definition 3.1.1 (*Theory of* C$^\infty$*-rings*)**.** The algebraic theory $\mathbb{T}_{C^\infty}$ of C$^\infty$**-Rings** over the signature Σ is defined as follows.

- Σ has one sort A
- For every $n \in \mathbb{N}$ and every smooth function $f : \mathbb{R}^n \to \mathbb{R}$ the signature Σ has an n-ary function symbol
$$f : \underbrace{A \cdots A}_{n} \to A$$
(The constants are precisely the real numbers $r \in \mathbb{R}$.)
- Σ has no relation symbols

The axioms of $\mathbb{T}_{C^\infty}$ are

(1) For $n, m \in \mathbb{N}$ and smooth functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g_1, \ldots, g_n : \mathbb{R}^m \to \mathbb{R}$ there is an axiom
$$\top \ \vdash_{\vec{x}} \ f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m)) = (f \circ (g_1, \ldots, g_n))(x_1, \ldots, x_m)$$

(2) (**Projection**) For every $n \geq 1$ and $1 \leq j \leq n$
$$\top \ \vdash_{\vec{x}} \ \mathrm{pr}_j(x_1, \ldots, x_n) = x_j$$
where $\mathrm{pr}_j : \mathbb{R}^n \to \mathbb{R}$ is the projection onto the jth factor.

Remark 3.1.2. The category of $\mathbb{T}_{C^\infty}$-models in a finite-product category C is equivalent to the category of L-models as categories over C. This can be seen from the comparison theorems in chapter 1.4. Indeed, L-models are equivalent to O_L-algebras for the endomorphism clone $O_L = \mathrm{End}(\mathbb{R})$ with $O_L(n) = L(\mathbb{R}^n, \mathbb{R}) = C^\infty(\mathbb{R}^n, \mathbb{R})$, and the latter are equivalent to models of the algebraic theory $\mathbb{T}_{O_L}$. But $\mathbb{T}_{O_L}$ is $\mathbb{T}_{C^\infty}$ as can be seen from the proof of theorem 1.4.1. We will write C^∞-Rng(C) for the category of $\mathbb{T}_{C^\infty}$-models in C, and C^∞-Rng for C^∞-Rng(Set).

All the operations making $\mathbb{R}$ into an $\mathbb{R}$-algebra are smooth maps. Every C^∞-ring R is thus an $\mathbb{R}$-algebra; in particular, so is the universal C^∞-ring A in the syntactic category $C_{\mathbb{T}_{C^\infty}}$. This yields a finite-limit preserving functor $F_A : C_{\mathbb{T}} \to C_{\mathbb{T}_{C^\infty}}$ mapping the universal $\mathbb{R}$-algebra to A and thus, by corollary 2.3.3, a functor $(F_A)_* : \mathrm{IAff}(C_{\mathbb{T}}) \to \mathrm{IAff}(C_{\mathbb{T}_{C^\infty}})$. It is easy to see that F_A and the induced $(F_A)_*$ are embeddings.

This shows that every 'space' in $C_{\mathbb{T}_{C^\infty}}$, which is the zero-locus of finitely many polynomials, carries a nil-square i-structure and is an i-affine space. Moreover, all morphisms that are definable by finitely many polynomials are i-affine maps. We wish to extend this result to all objects corresponding to Horn formulae and all morphisms in $C_{\mathbb{T}_{C^\infty}}$. The key lemma that makes this work (and also allows us to give essentially the same proof as in the case of K-algebras) is *Hadamard's lemma*.

Before we state Hadamard's lemma, we introduce some notation. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. We denote its kth derivative by $\partial^k f$. It is a smooth map

$$\partial^k f : \mathbb{R}^n \to \mathcal{L}^k(\mathbb{R}^n, \mathbb{R}),$$

where $\mathcal{L}^k(\mathbb{R}^n, \mathbb{R})$ denotes the $\mathbb{R}$-vector space of k-linear forms $\underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{k} \to \mathbb{R}$. The evaluation of the k-linear form $\partial^k f(x)$ at the vectors $h_1, \ldots, h_k$ is denoted by $\partial^k f(x)[h_1, \ldots, h_k]$, and if all the h_j are equal we use the shorthand $\partial^k f(x)[h]^k$.

Lemma 3.1.3 (Hadamard's Lemma)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. There is a smooth map $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}^2(\mathbb{R}^n, \mathbb{R})$ such that*

$$f(\vec{y}) - f(\vec{x}) = \partial f(\vec{x})[\vec{y} - \vec{x}] + g(\vec{x}, \vec{y})[\vec{y} - \vec{x}]^2$$

Proof. This is a simple consequence of Taylor's formula with integral remainder [AE08, thm. II.5.8]. Indeed, by Taylor's formula of order 2 we have

$$\begin{aligned} f(\vec{x}+\vec{h}) &= f(\vec{x}) + \partial f(\vec{x})[\vec{h}] + \frac{1}{2}\partial^2 f(\vec{x})[\vec{h}] + \int_0^1 (1-t)(\partial^2 f(\vec{x}+t\vec{h}) - \partial^2 f(\vec{x}))[\vec{h}]^2 dt \\ &= f(\vec{x}) + \partial f(\vec{x})[\vec{h}] + \int_0^1 (1-t)\partial^2 f(\vec{x}+t\vec{h})[\vec{h}]^2 dt \end{aligned}$$

We define g by the vector-valued integrals

$$g(\vec{x},\vec{y}) = \int_0^1 (1-t)\partial^2 f((1-t)\vec{x}+t\vec{y})dt$$

This is clearly a smooth map. Since

$$\Big(\int_0^1 (1-t)\partial^2 f((1-t)\vec{x}+t\vec{y})dt\Big)[h_1,h_2] = \int_0^1 (1-t)\partial^2 f((1-t)\vec{x}+t\vec{y})[h_1,h_2]dt,$$

it is also bilinear. Substituting $\vec{h} = \vec{y} - \vec{x}$ gives the desired formula. □

Remarks 3.1.4.

(a) If we use the canonical basis of $\mathbb{R}^n$ then Hadamard's lemma can be equivalently stated as: there are n smooth functions $a_i : \mathbb{R}^n \to \mathbb{R}$, and n^2 smooth functions $g_{ij} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$f(\vec{y}) - f(\vec{x}) = \sum_{i=1}^{n} a_i(\vec{x})(y_i - x_i) + \sum_{i,j=1}^{n} g_{ij}(\vec{x},\vec{y})(y_i - x_i)(y_j - x_j)$$

Since $\partial^2 f$ factors through the $\mathbb{R}$-vector space of symmetric bilinear forms, each $g(\vec{x},\vec{y})$ as constructed in the proof is symmetric; in particular, we may choose $g_{ij} = g_{ji}$.

(b) Taylor's formula allows us to extend Hadamard's lemma to higher orders easily: for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$ and $\ell \geq 1$ there is a smooth function $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}^\ell(\mathbb{R}^n, \mathbb{R})$ such that
$$f(\vec{y}) - f(\vec{x}) = \sum_{k=1}^{\ell-1} \frac{1}{k!}\partial^k f(\vec{x})[\vec{y} - \vec{x}]^k + g(\vec{x}, \vec{y})[\vec{y} - \vec{x}]^\ell$$
Lemma 3.1.3 covers the case $\ell = 2$. The case $\ell = 1$ is what is usually referred to as Hadamard's lemma in the literature.

Theorem 3.1.5 (Syntactic smooth i-affine spaces)**.** *Let $C_{\mathbb{T}_{C^\infty}}$ denote the syntactic category of the algebraic theory $\mathbb{T}_{C^\infty}$ of C^∞-rings, and let $\mathring{C}_{\mathbb{T}_{C^\infty}}$ denote the full subcategory generated by objects corresponding to Horn formulae. The forgetful functor $U : \mathrm{IAff}(C_{\mathbb{T}_{C^\infty}}) \to C_{\mathbb{T}_{C^\infty}}$ has a section $Af : \mathring{C}_{\mathbb{T}_{C^\infty}} \to \mathrm{IAff}(C_{\mathbb{T}_{C^\infty}})$, which maps each object $\{\vec{x}.\phi\}$ to the i-affine space induced by the nil-square i-structure over $\{\vec{x}.\phi\}$.*

Proof. The proof is essentially the same as the combined proofs of theorem 2.1.5 and theorem 2.3.1.

(1) As usual, A^n denotes $\{\vec{x}.\top\}$, where n is the length of the context $\vec{x}$. Because of the embedding $(F_A)_* : \mathrm{IAff}(C_\mathbb{T}) \to \mathrm{IAff}(C_{\mathbb{T}_{C^\infty}})$ we know that A^n is an i-affine space, where the i-structure and the i-affine structure are defined as in the case of K-algebras. The nil-square i-structure on A^n is thus the i-structure generated by the symmetric, reflexive binary relation $\partial_n \equiv \bigwedge_{1 \leq i,j \leq n}(x_i - y_i)(x_j - y_j) = 0$:
$$A^n\langle m\rangle = \{\vec{x}^1, \ldots, \vec{x}^m. \bigwedge_{1 \leq k,\ell \leq m} \partial_n[\vec{x}^k, \vec{x}^\ell/\vec{x}, \vec{y}]\}$$
and the action $\bullet_m : \mathcal{A}(m) \times A^n\langle m\rangle \to A^n$ by the (embedded) clone of affine combinations $\mathcal{A}$ is given by the $\mathbb{T}$-provable equivalence class of the formula-in-context
$$\vec{\lambda}, \vec{x}^1, \ldots, \vec{x}^m, \vec{y}.(\vec{y} = \lambda_1\vec{x}^1 + \ldots + \lambda_m\vec{x}^m) \wedge \alpha_m[\vec{\lambda}/\vec{x}] \wedge \bigwedge_{1 \leq k,\ell \leq m} \partial_n[\vec{x}^k, \vec{x}^\ell/\vec{x}, \vec{y}]$$
where α_m is the formula $1 = x_1 + \ldots + x_m$. We only need to show that the i-structure and i-affine structure restrict to the subobjects $\{\vec{x}.\phi\} \rightarrowtail A^n$ for a cartesian formula ϕ over the signature of the theory of C^∞-rings.

(2) For a Horn formulae ϕ, the nil-square i-structure and i-affine structure on $\{\vec{x}.\phi\}$ are also defined in the same way as for K-algebras
$$\{\vec{x}.\phi\}\langle m\rangle = \{\vec{x}^1, \ldots, \vec{x}^m. \bigwedge_{1 \leq k,\ell \leq m} \partial_n[\vec{x}^k, \vec{x}^\ell/\vec{x}, \vec{y}] \wedge \bigwedge_{1 \leq k \leq m} \phi[\vec{x}^k/\vec{x}]\}$$

For each $m \in \mathbb{N}$ the morphism $\bullet_m : \mathcal{A}(m) \times \{\vec{x}.\phi\}\langle m\rangle \to \{\vec{x}.\phi\}$ is represented by the formula-in-context

$$\vec{\lambda}, \vec{x}^1, \dots, \vec{x}^m, \vec{y}.(\vec{y} = \sum_{k=1}^{m} \lambda_k \vec{x}^k) \wedge \alpha_m[\vec{\lambda}/\vec{x}] \wedge \bigwedge_{1\le k,\ell\le m} \partial_n[\vec{x}^k, \vec{x}^\ell/\vec{x}, \vec{y}] \wedge \bigwedge_{1\le k\le m} \phi[\vec{x}^k/\vec{x}]$$

Clearly, these definitions make the inclusion $\{\vec{x}.\phi\} \rightarrowtail A^n$ an i-structure reflecting i-affine map, as long as we can show that $\bullet_m$ is well-defined. This amounts to show the sequent

$$(\vec{y} = \sum_{k=1}^{m} \lambda_k \vec{x}^k) \wedge \alpha_m[\vec{\lambda}/\vec{x}] \wedge \bigwedge_{1\le k,\ell\le m} \partial_n[\vec{x}^k, \vec{x}^\ell/\vec{x}, \vec{y}] \wedge \bigwedge_{1\le k\le m} \phi[\vec{x}^k/\vec{x}] \vdash_{\vec{\lambda},\vec{x}^1,\dots,\vec{x}^m,\vec{y}} \phi[\vec{y}/\vec{x}]$$

provable in $\mathbb{T}_{C^\infty}$, and we can show this in the same way as we did for K-algebras. Firstly, we can replace $\vec{x}.\phi$ by its $\mathbb{T}_{C^\infty}$-provably equivalent normal form $\vec{x}.\psi$, where ψ is a conjunction of equations of the form $f(\vec{x}) = 0$ for smooth functions $f : \mathbb{R}^n \to \mathbb{R}$. The assertion follows from the sequent

$$\alpha_m[\vec{\lambda}/\vec{x}] \wedge \bigwedge_{1\le k,\ell\le m} \partial_n[\vec{x}_k, \vec{x}_\ell/\vec{x}, \vec{y}] \vdash_{\vec{\lambda},\vec{x}_1,\dots,\vec{x}_m} f\big(\sum_{k=1}^{m} \lambda_k \vec{x}_k\big) = \sum_{k=1}^{m} \lambda_k f(\vec{x}_k)$$

stating that smooth maps preserve i-affine combinations, which is $\mathbb{T}_{C^\infty}$-provable due to Hadamard's lemma. Indeed, we can write

$$f(\vec{x}_\ell) - f(\vec{x}) = \partial f(\vec{x})[\vec{x}_\ell - \vec{x}] + g(\vec{x}, \vec{x}_\ell)[\vec{x}_\ell - \vec{x}]^2$$

(This is due to axiom (1) and (2) of $\mathbb{T}_{C^\infty}$, remark 3.1.4(b), and the fact that multiplication, addition and subtraction are all smooth maps.) If $\vec{x}_\ell \sim_1 \vec{x}$ (i.e. $\partial_n[\vec{x}_\ell, \vec{x}/\vec{x}, \vec{y}]$) then the equation simplifies to

$$f(\vec{x}_\ell) - f(\vec{x}) = \partial f(\vec{x})[\vec{x}_\ell - \vec{x}]$$

We know that $\vec{x}_\ell \sim_1 \sum_{k=1}^m \lambda_k \vec{x}_k$. Therefore, by using that $\partial f(\vec{x})$ is $\mathbb{R}$-linear and $\sum_{\ell=1}^m \lambda_\ell = 1$ we compute

$$\begin{aligned}
\sum_{\ell=1}^m \lambda_\ell f(\vec{x}_\ell) - f\Big(\sum_{k=1}^m \lambda_k \vec{x}_k\Big) &= \sum_{\ell=1}^m \lambda_\ell \Big(f(\vec{x}_\ell) - f\Big(\sum_{k=1}^m \lambda_k \vec{x}_k\Big)\Big) \\
&= \sum_{\ell=1}^m \lambda_\ell \partial f\Big(\sum_{k=1}^m \lambda_k \vec{x}_k\Big)\Big[\vec{x}_\ell - \sum_{k=1}^m \lambda_k \vec{x}_k\Big] \\
&= \partial f\Big(\sum_{k=1}^m \lambda_k \vec{x}_k\Big)\Big[\sum_{\ell=1}^m \lambda_\ell \vec{x}_\ell - \sum_{\ell=1}^m \lambda_\ell \sum_{k=1}^m \lambda_k \vec{x}_k\Big] \\
&= \partial f\Big(\sum_{k=1}^m \lambda_k \vec{x}_k\Big)\Big[\sum_{\ell=1}^m \lambda_\ell \vec{x}_\ell - \sum_{k=1}^m \lambda_k \vec{x}_k\Big] \\
&= 0
\end{aligned}$$

(3) It remains to show that every morphism in $\mathring{C}_{\mathbb{T}_{C^\infty}}$ is i-affine. For this purpose we consider the equivalent category C$^\infty$-Rng$_{fp}^{op}$, i.e. the opposite category of finitely-presented C$^\infty$-rings in Set [Cos76, III.c]. The objects of C$^\infty$-Rng$_{fp}$ are all presentations of the form $C^\infty(\mathbb{R}^n)/(f_1, \ldots, f_m)$ for some $n, m \in \mathbb{N}$. These presentations can be interpreted as C$^\infty$-rings by forming the respective quotient $\mathbb{R}$-algebra, where $(f_1, \ldots, f_m)$ is taken as the ($\mathbb{R}$-algebra) ideal generated by the $f_i \in C^\infty(\mathbb{R}^n)$. Indeed, the $\mathbb{R}$-algebra $C^\infty(\mathbb{R}^n) = C^\infty(\mathbb{R}^n, \mathbb{R})$ is the free C$^\infty$-ring on n generators, which are the the projections $\mathrm{pr}_i : \mathbb{R}^n \to \mathbb{R}$. (Cf. [MR91, prop. 1.1] and proposition 1.3.12) From Hadamard's lemma it follows that the $\mathbb{R}$-algebra $C^\infty(\mathbb{R}^n)/I$ is a C$^\infty$-ring and the canonical quotient map a C$^\infty$-ring homomorphism [MR91, prop. 1.2]. Due to proposition 1.1.21 we may conclude that every finitely presented $\mathbb{T}_{C^\infty}$-model is of the asserted form.

That each morphism in $\mathring{C}_{\mathbb{T}_{C^\infty}}$ is an i-affine map follows as in the proofs of theorem 2.1.5 and part (3) of the proof of theorem 2.3.1. We only have to replace the polynomial rings $K[X_1, \ldots, X_n]$ by $C^\infty(\mathbb{R}^n)$, the generators X_i by the projections pr_i, the tensor product '$\otimes_K$' by the coproduct '$\otimes_\infty$', and apply the smooth version of Hadamard's lemma instead of the polynomial version.

□

Corollary 3.1.6. *Let C be a category with finite limits, and R a* C^∞*-ring object in C. Let $F_R : C_{\mathbb{T}_{C^\infty}} \to C$ be the functor that maps the universal* C^∞*-ring A to R. We have*

$$\begin{tikzcd} \mathrm{IAff}(C_{\mathbb{T}_{C^\infty}}) \arrow[r, "(F_R)_*"] & \mathrm{IAff}(C) \arrow[d, "U"] \\ \mathring{C}_{\mathbb{T}_{C^\infty}} \arrow[u, hook, "Af"] \arrow[r, "F_R"] & C \end{tikzcd}$$

where $(F_R)_*$ is the functor induced by F_R as in proposition 2.2.6(2)*.*

Remark 3.1.7 (Cartesian Existential Quantifiers)**.** As in the case of K-algebras the result of theorem 3.1.5 can be extend to certain cartesian formulae-in-context over the signature of $\mathbb{T}_{C^\infty}$.

Let $\vec{x}.(\exists!\vec{y})\phi$ be a cartesian formula-in-context, where ϕ is a conjunction $(f_1(\vec{x},\vec{y}) = 0)\wedge\ldots\wedge(f_\ell(\vec{x},\vec{y}) = 0)$, n is the length of the context $\vec{x}$, m is the length of $\vec{y}$ and $f_j \in C^\infty(\mathbb{R}^n\times\mathbb{R}^m)$. Since every C^∞-ring is an $\mathbb{R}$-algebra we can essentially repeat the proof of lemma 2.1.10. As before, we only need to replace the polynomial K-algebras in n indeterminates $K[\vec{x}]$ with the free C^∞-rings of n generators $C^\infty(\mathbb{R}^n)$ and apply the smooth version of Hadamard's lemma. (The quotients of C^∞-rings are the quotients of the underlying $\mathbb{R}$-algebras.) This yields once more that the system of equations given by ϕ is overdetermined at worst, i.e. $m \le \ell$. The proof of theorem 2.1.11 can be adapted in the same way, and we obtain that for $m = \ell$ the $C_{\mathbb{T}_{C^\infty}}$-isomorphism

$$f : \{\vec{x},\vec{y}.\phi\} \to \{\vec{x}'.(\exists!\vec{y})\phi[\vec{x}'/\vec{x}]\}$$

represented by $\vec{y},\vec{x},\vec{x}'.\phi\wedge(\vec{x}=\vec{x}')$ is an isomorphism of nil-square i-structures. Moreover, due to the smooth Hadamard's lemma the proof of theorem 2.3.2 carries over as well and we see that f is an isomorphism of i-affine spaces.

The proof of the overdetermined case of theorem 2.1.12 cannot be adopted and rephrased for the smooth case in a straight-forward manner. Here a different approach is needed.

Let Mfd denote the category of smooth (paracompact) manifolds and smooth maps. There is a fully faithful embedding [MR91, chap. I.2]

$$\mathrm{Mfd} \rightarrowtail C^\infty\text{-}\mathrm{Rng}^{op}, \quad M \mapsto C^\infty(M,\mathbb{R})$$

$C^\infty(M,\mathbb{R})$ is clearly an algebra for the endomorphism clone $\mathrm{End}(\mathbb{R})$ of smooth functions, and hence a C^∞-ring. In fact, $C^\infty(M,\mathbb{R})$ is finitely presentable. To see this, we use Whitney's embedding theorem together with the fact that every submanifold M of $\mathbb{R}^n$ is a smooth retract of an open subset U (a normal neighbourhood of M). Since $C^\infty(U)$ is finitely presentable [MR91, cor. I.2.2], retracts of finitely presentable objects (in the sense of category theory [Joh02, def. D2.3.1]) are finitely presentable, and since both notions coincide [Joh02, chap. D2.4]), $\mathrm{C}^\infty(M)$ is finitely presentable. By fixing a presentation for each manifold, the embedding $\mathrm{C}^\infty(-,\mathbb{R})$ factors through the subcategory $\mathrm{C}^\infty\text{-Rng}^{op}_{fp} \simeq \mathring{C}_{\mathbb{T}_{\mathrm{C}^\infty}}$ up to a natural isomorphism. (Choosing different presentations yields isomorphic embeddings into $C_{\mathbb{T}_{\mathrm{C}^\infty}}$.)

Corollary 3.1.8. *Under the embedding* $\mathrm{Mfd} \rightarrowtail \mathring{C}_{\mathbb{T}_{\mathrm{C}^\infty}}$ *(determined up to a natural isomorphism) every smooth (paracompact) manifold admits the structure of an i-affine space over its nil-square i-structure, and every smooth map between manifolds becomes an i-affine map, when embedded into* $\mathring{C}_{\mathbb{T}_{\mathrm{C}^\infty}}$.

Remark 3.1.9. The corollary implies that every submanifold of $\mathbb{R}^n$ is an i-affine subspace of $\mathbb{R}^n$ and the inclusion reflects i-structure. From the geometric viewpoint, however, rather than using Whitney's embedding theorem, we would like to construct the i-affine structure on a manifold by gluing together the i-affine structures on the open subsets of its atlas. We shall study this in section 3.2.2.

As in the case of K-algebras we can extend the result from $\mathrm{C}^\infty\text{-Rng}^{op}_{fp}$ to $\mathrm{C}^\infty\text{-Rng}^{op}$.

Theorem 3.1.10. *There is an extension of* $Af : \mathrm{C}^\infty\text{-Rng}^{op}_{fp} \to \mathrm{IAff}(\mathrm{C}^\infty\text{-Rng}^{op}_{fp})$ *to a functor* $Af : \mathrm{C}^\infty\text{-Rng}^{op} \to \mathrm{IAff}(\mathrm{C}^\infty\text{-Rng}^{op})$*, i.e. the subsequent diagram commutes for the fully faithful functor* $\iota : K\text{-Alg}^{op}_{fp} \to K\text{-Alg}^{op}$

$$\begin{tikzcd}
\mathrm{C}^\infty\text{-Rng}^{op} \arrow[r, "Af"] & \mathrm{IAff}(\mathrm{C}^\infty\text{-Rng}^{op}) \\
\mathrm{C}^\infty\text{-Rng}^{op}_{fp} \arrow[u, "\iota"] \arrow[r, "Af"] & \mathrm{IAff}(\mathrm{C}^\infty\text{-Rng}^{op}_{fp}) \arrow[u, "\iota_*"']
\end{tikzcd}$$

and we have a natural isomorphism $U \circ Af \cong I_{\mathrm{C}^\infty\text{-Rng}^{op}}$*, where* $U : \mathrm{IAff}(\mathrm{C}^\infty\text{-Rng}^{op}) \to \mathrm{C}^\infty\text{-Rng}^{op}$ *denotes the forgetful functor. (Note that since we work with finitely presented* C^∞*-rings instead of finitely presentable ones,* ι *is not injective on objects, and hence not an embedding.)*

Proof. We have two ways to proof this. One way is to repeat the same proof as for theorem 2.3.5 and make substitutions with the respective C^∞-ring analogues of the K-algebra constructions. Indeed, the free C^∞-ring $F_{\mathbb{T}_{\mathrm{C}^\infty}}(X)$ over the set X underlies the set of smooth functions of finitely many variables, which are indexed by the finite subsets of X. This is due to the fact that filtered colimits of C^∞-rings in Set are the filtered colimits of their underlying sets (as it is the case for any Set-models of a cartesian theory).

However, we also know that the functor ι is *dense*, and every C^∞-ring A is a filtered colimit of finitely presented C^∞-rings. (The diagram category and colimit cone can be chosen canonically as the comma category $(\iota \downarrow A)$ and the C^∞-ring homomorphisms $f : \iota(B) \to A$ for every finitely C^∞-ring B and C^∞-ring homomorphism f. Cf. [Joh02, lem. D2.3.2], for example.) This means that in $\mathrm{C}^\infty\text{-Rng}^{op}$ the object A is a canonical cofiltered limit of finitely presented C^∞-rings in $\mathrm{C}^\infty\text{-Rng}^{op}_{fp}$. Now it is easy to see that the proof of theorem 2.6.4 still applies, if we replace the Grothendieck topos $\mathcal{S}$ with a complete category $\mathcal{S}$, and hence that the forgetful functor $U : \mathrm{IAff}(\mathrm{C}^\infty\text{-Rng}^{op}) \to \mathrm{C}^\infty\text{-Rng}^{op}$ lifts limits uniquely. This allows us to construct the desired extension of Af along ι. (Note that with this construction the extension Af is a section of the forgetful functor U, i.e., $U \circ Af = I_{\mathrm{C}^\infty\text{-Rng}^{op}}$.) □

3.2 Well-adapted models

A *model for SDG* is a topos $\mathcal{S}$ together with a K-algebra object R of **line type**. The latter means that R satisfies an axiom schema of Kock-Lawvere type. The axioms state that for all *infinitesimal subspaces* D of R^n, all maps $f : D \to R$ are polynomial, and the polynomials are uniquely determined by f. (Here we are using the internal higher-order language of the topos $\mathcal{S}$ and quantifiy over $[D, R]$, not just the set $\mathcal{S}(D, R)$. Cf. [Joh02, D4].)

An infinitesimal subspace D is defined as the (R-)interpretation of the formula-in-context $d_1, \cdots, d_k.\ \bigwedge_{1\le j\le k} d_j^{n_j} = 0 \wedge \phi$, where each $n_j \ge 1$ and ϕ is a Horn formula over the signature of K-algebras. In the internal higher-order language of $\mathcal{S}$ the corresponding Kock-Lawvere axiom states that for every $f \in [D, R]$ there are $a_\alpha \in R$ indexed (externally) by multi-indices $\alpha \in \mathbb{N}^k$ such that

$$f(d_1, \dots, d_k) = \sum_{|\alpha|\le n} a_\alpha d^\alpha$$

where n is the sum of the n_j, $|\alpha| = \alpha_1 + \dots + \alpha_k$, and d^α stands for $d_1^{\alpha_1} \dots d_k^{\alpha_k}$. The a_α, for which d^α is not the zero map on D, are uniquely determined by f.

In the literature one uses the corresponding *finitely presentable* K-algebras instead of the formulae-in-context to define infinitesimal spaces. (Recall that the syntactic category of K-algebras and $K\text{-Alg}^{op}_{fp}$ are equivalent as categories. Mapping each presentation to the respective quotient algebra yields a fully faithful isomorphism dense functor from $K\text{-Alg}_{fp}$ to the category $K\text{-Alg}_\omega$ of finitely presentable K-algebras. This becomes an equivalence of categories by invoking the axiom of choice.) These algebras can be characterised as K-algebras W that are finitely generated and free as K-modules, are local, and admit a decomposition $W \cong K1 \oplus I$ as K-modules, where M consists of nil-potent elements. Such algebras are called

Weil algebras over K (cf. [Joh13, def. F1.1.9]). The infinitesimal space corresponding to W is denoted by D_W and considered as the R-spectrum of W. One can construct a natural K-algebra homomorphism $\alpha_W : R \otimes_K W \to R^{D_W}$, and the corresponding Kock-Lawvere axiom sates that this α_W is an isomorphism. (See [Koc81], or [Joh13, F1].)

The notion of *well-adapted model* is a model of SDG that takes the category Mfd of smooth manifolds into account. It has been introduced by Dubuc in [Dub79] and further simplified by Kock in [Koc81]. Our definition follows [Joh13], which drops the requirement that the infinitesimal spaces D_W are *atoms*, i.e. $(-)^{D_W}$ preserves colimits, but demands the functor ι to be a fully faithful embedding. Furthermore, we shall only consider well-adapted models in Grothendieck toposes here.

Definition 3.2.1 (*Well-adapted model*)**.** A **well-adapted model** $(\mathcal{S}, \iota)$ is a Grothendieck topos $\mathcal{S}$ together with a fully faithful embedding $\iota : \mathrm{Mfd} \to \mathcal{S}$ satisfying

(1) The functor ι preserves the terminal object and *transversal pullbacks*, i.e. pullbacks of cospans $M \xrightarrow{f} Z \xleftarrow{g} N$, where for each $p \in M$ and $q \in N$ such that $x = f(p) = g(q)$ the sum of the $\mathbb{R}$-vector subspaces $\operatorname{im} T_p f$ and $\operatorname{im} T_q g$ is $T_x N$.

(2) The functor ι maps open covers to jointly epimorphic families.

(3) $R = \iota(\mathbb{R})$ is of *line type*.

Remark 3.2.2. $\mathbb{R}$ is a C^∞-ring in Mfd. (This is because C^∞-rings are $\mathrm{End}(\mathbb{R})$-algebras of the endomorphism clone $\mathrm{End}(\mathbb{R})$ for $\mathbb{R}$ in Mfd. More explicitly, the operation for a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ is given by evaluation on $\mathbb{R}^n$.) Since the embedding ι preserves transversal pullbacks and the terminal object, it preserves finite products, in particular. This shows $\iota(\mathbb{R})$ a C^∞-ring.

In fact, Dubuc and Bunge have shown in [BD86] that there is a one-to-one correspondence between (isomorphism classes of) functors $i : \mathrm{Mfd} \to \mathcal{S}$ (not necessarily a fully faithful embedding) satisfying the axioms (1) and (2) of a well-adapted model, and *archimedean local* C^∞-rings in $\mathcal{S}$. A C^∞-ring A is called local, if it is non-trivial and satisfies the sequent

$$(\exists z)(x + y)z = 1 \ \vdash_{x,y} \ ((\exists z)xz = 1) \vee ((\exists z)yz = 1)$$

It is called archimedean if it has an ireflexive, transitive relation '>' compatible with the ring structure and satisfies the sequents

$$
\begin{aligned}
x+y>0 &\vdash_{x,y} (x>0)\vee(y>0)\\
(\exists y)xy=1 &\vdash_{x} (x>0)\vee(0>x)\\
\top &\vdash_{x} \bigvee_{n\in\mathbb{N}}(n1>x)
\end{aligned}
$$

Every local C^∞-ring in a finite limit category admits a unique such strict order relation '>' compatible with the ring structure and satisfying the first two sequents [Joh13, prop. F1.5.3].

3.2.1 The classifying topos of archimedean local C^∞-rings

Let $C^\infty\text{-Rng}_\omega$ denote the category of *finitely presentable* (not finitely presented) C^∞-rings. The *classifying topos* of archimedean local C^∞-rings can be constructed as the topos $\mathrm{Sh}(C^\infty\text{-Rng}_\omega^{op}, D)$ of sheaves on the site[1] $(C^\infty\text{-Rng}_\omega^{op}, D)$, where D stands for the *Dubuc coverage*. The D-covering families on a C^∞-ring A are (small) families of epimorphisms $(A \twoheadrightarrow A[a_i^{-1}])_{i\in I}$ in $C^\infty\text{-Rng}_\omega$ such that for each C^∞-ring homomorphism (in fact, for each $\mathbb{R}$-algebra homomorphism) $p : A \to \mathbb{R}$ there is at least one $i \in I$ such that $p(a_i) \neq 0$. Here $A[a_i^{-1}]$ denotes the C^∞-localisation of A at $a_i \in A$. The Dubuc coverage is *subcanonical*; hence the yoneda embedding of $C^\infty\text{-Rng}_\omega^{op}$ factors through $\mathrm{Sh}(C^\infty\text{-Rng}_\omega^{op}, D)$. The fully faithful embedding

$$
i : \mathrm{Mfd} \xrightarrow{C^\infty(-,\mathbb{R})} C^\infty\text{-Rng}_\omega^{op} \xrightarrow{\;y\;} \mathrm{Sh}(C^\infty\text{-Rng}_\omega^{op}, D)
$$

preserves transversal pullbacks and the terminal object. This is because $C^\infty(-,\mathbb{R})$ preserves these types of finite limits, and y preserves all finite limits. The Dubuc coverage restricts to open covers in Mfd along the embedding $C^\infty(-,\mathbb{R})$, and y maps Dubuc covers to jointly epimorphic families; hence i maps open covers to jointly epimorphic families. Moreover, $i(\mathbb{R})$ is of line type. This shows $(\mathrm{Sh}(C^\infty\text{-Rng}_\omega^{op}, D), i)$ a well-adapted model. (For the proofs of these statements we refer to [Joh13, chap. F1.3].)

Since $C^\infty\text{-Rng}_\omega$ is a subcategory of C^∞-Rng we can apply theorem 3.1.10 (respectively corollary 3.1.6) to the finite-limit preserving functor y to get that every representable sheaf is an i-affine space over its nil-square i-structure. In fact, for every C^∞-Ring A, the hom-functor $C^\infty\text{-Rng}(A, -)$ restricts to a D-sheaf [Joh13, prop. F1.3.23]. The yoneda embedding has thus an extension to a finite-limit preserving embedding $\bar{y} : C^\infty\text{-Rng}^{op} \to \mathrm{Sh}(C^\infty\text{-Rng}_\omega^{op}, D)$ and hence

[1] For the notion of *coverage*, *site* and *sheaf* we refer to [Joh02, chap. C2].

yields an embedding

$$\bar{y}_* : \mathrm{IAff}(\mathrm{C}^\infty\text{-}\mathrm{Rng}^{op}) \to \mathrm{IAff}(\mathrm{Sh}(\mathrm{C}^\infty\text{-}\mathrm{Rng}^{op}_\omega, D))$$

The embedding i factorises as

$$i : \mathrm{Mfd} \xrightarrow{\mathrm{C}^\infty(-,\mathbb{R})} \mathrm{C}^\infty\text{-}\mathrm{Rng}^{op} \xrightarrow{\bar{y}} \mathrm{Sh}(\mathrm{C}^\infty\text{-}\mathrm{Rng}^{op}_\omega, D)$$

Now, given a well-adapted model $(\mathcal{S}, \iota)$ we know that $\iota(\mathbb{R})$ is an archimedean local C^∞-ring; hence there is a geometric morphism $F : \mathcal{S} \to \mathrm{Sh}(\mathrm{C}^\infty\text{-}\mathrm{Rng}^{op}_\omega, D)$ (unique up to unique natural isomorphism) such that $F^*(i(\mathbb{R})) = \iota(\mathbb{R})$. Moreover, the subsequent diagram commutes up to a natural isomorphism

$$\begin{tikzcd}
\mathcal{S} & \mathrm{Sh}(\mathrm{C}^\infty\text{-}\mathrm{Rng}^{op}_\omega, D) \arrow[l, "F^*"'] \\
\mathrm{Mfd} \arrow[u, tail, "\iota"] \arrow[r, tail, "\mathrm{C}^\infty(-,\mathbb{R})"] & \mathrm{C}^\infty\text{-}\mathrm{Rng}^{op} \arrow[u, tail, "\bar{y}"']
\end{tikzcd}$$

The inverse-image functor F^* preserves finite limits and colimits, so it induces a functor

$$(F^*)_* : \mathrm{IAff}(\mathrm{Sh}(\mathrm{C}^\infty\text{-}\mathrm{Rng}^{op}_\omega, D)) \to \mathrm{IAff}(\mathcal{S})$$

Moreover, any i-affine spaces constructed from colimits in $\mathrm{Sh}(\mathrm{C}^\infty\text{-}\mathrm{Rng}^{op}_\omega, D)$, which the forgetful functor $U : \mathrm{IAff}(\mathrm{Sh}(\mathrm{C}^\infty\text{-}\mathrm{Rng}^{op}_\omega, D)) \to \mathrm{Sh}(\mathrm{C}^\infty\text{-}\mathrm{Rng}^{op}_\omega, D)$ lifts, also carry over to respective colimit constructions of i-affine spaces in $\mathcal{S}$. This shows that well-adapted models come with a good supply of i-affine spaces. In particular, we have:

Theorem 3.2.3. *In a well-adapted model $(\mathcal{S}, \iota)$ every manifold $\iota(M)$ is an i-affine space (over the $\mathbb{R}$-algebra $\iota(\mathbb{R})$) with nil-square i-structure, and every morphism $f : \iota(M) \to \iota(N)$ is i-affine.*

Remarks 3.2.4.

(a) We have not needed that $\iota(\mathbb{R})$ is of line type to establish that every manifold $\iota(M)$ is i-affine; only the existence of an archimedean local C^∞-ring R in the Grothendieck topos $\mathcal{S}$ and that ι is full was needed. If we assume an archimedean local C^∞-ring R only, we have a geometric morphism $F : \mathcal{S} \to \mathrm{Sh}(\mathrm{C}^\infty\text{-}\mathrm{Rng}^{op}_\omega, D)$ such that $F^*(i(\mathbb{R})) = R$. In particular, we have a functor $\iota : \mathrm{Mfd} \to \mathcal{S}$ (not necessarily a fully faithful embedding) preserving transversal pullbacks, the terminal object, and mapping open covers to jointly epimorphic families. (By the result of Bunge and Dubuc this functor is unique up to isomorphism.)

Since F^* preserves finite-limits and colimits we obtain a supply of i-affine spaces in $\mathcal{S}$; in particular, all the spaces $\iota(M)$ are i-affine with nil-square i-structure and smooth maps $\iota(f)$ are i-affine maps.

If R is of line type (or, at least satisfies a certain smaller set of the Kock-Lawvere axioms), then we shall see in section 3.2.2 that every morphism $f : \iota(M) \to \iota(N)$ is i-affine, even if ι is not full.

(b) There are many models of SDG (not necessarily well-adapted) that can be constructed as toposes $\mathrm{Sh}(C^{op}, J)$ where C is is a subcategory of C^∞-Rng and J is subcanonical. (Cf. [MR91, appendix 2]) In particular, C can properly contain the category $\mathrm{C}^\infty\text{-Rng}_\omega$; for example, if we take C to be the full subcategory of C^∞-rings of the form $\mathrm{C}^\infty(\mathbb{R}^n)/I$ where I is a *germ-determined* ideal. (See [Joh13, def. F1.3.17] and [MR91, chap. III.1].) For such sites, the yoneda embedding of C^{op} factors through $\mathrm{Sh}(C^{op}, J)$, and we equally obtain a fully faithful embedding $(y)_* : \mathrm{IAff}(C^{op}) \to \mathrm{IAff}(\mathrm{Sh}(C^{op}, J))$.

3.2.2 Formal manifolds

The main aim of this section is to show that *formal manifolds* in SDG are i-affine spaces. Since smooth manifolds $\iota(M)$ are formal manifolds in a well-adapted model $(\mathcal{S}, \iota)$, we fullfil the promise made earlier, and construct the i-affine structure of a manifold by gluing together the i-affine structures of the open subsets of its atlas. We shall also give an example of an i-affine space in SDG, that is not *infinitesimally smooth*. (In the literature infinitesimally smooth spaces are also referred to as microlinear spaces. A related, but slightly weaker notion is that of an infinitesimally linear space.) The subsequent definition is taken from [Joh13].

Definition 3.2.5 (*Formal manifolds*)**.** Let $\mathcal{S}$ be a Grothendieck topos, and R a K-algebra object of line type.

(1) A subobject $\iota : U \rightarrowtail M$ in $\mathcal{S}$ is called **formally open**, if the subsequent square

$$\begin{tikzcd}
[D_W, U] \arrow[r, tail, "{[D_W,\iota]}"] \arrow[d, "\mathrm{ev}_0"] & {[D_W, M]} \arrow[d, "\mathrm{ev}_0"] \\
U \arrow[r, tail, "\iota"] & M
\end{tikzcd}$$

is a pullback for every Weil algebra W. (Here ev_0 stand for the evaluation-at-0 morphism.)

(2) An object M in $\mathcal{S}$ is a said to be a **formal manifold** (of dimension n), if there is a covering family $\iota_j : U_j \rightarrowtail M$, $j \in I$, i.e. $M = \bigvee_{j \in I} U_j$, of formally open subobjects U_j, which are

also formally open subobjects of $\phi_j : U_j \rightarrowtail R^n$ of R^n . Each such (ι_j, ϕ_j) is called an (n dimensional) **chart** of M, and each covering family of charts is called an (n-dimensional) **atlas** for M.

Remarks 3.2.6.

(a) Since R is of line type, one can show that a subobject $\iota : U \rightarrowtail R^n$ is formally open if and only if it satisfies the pullback condition for the infinitesimal spaces $D_k(n) = \{\vec{d} \in R^n \mid d_{i_1} \cdots d_{i_{k+1}} = 0,\ 1 \leq i_1, \ldots, i_{k+1} \leq n\}$ for each $k \geq 1$ (i.e. $D_k(n) = D_{W(n,k)}$, where $W(n,k) = K[X_1, \ldots, X_n]/(X_{i_1} \cdots X_{i_{k+1}},\ 1 \leq i_1, \ldots, i_{k+1} \leq n)$). (See also [Joh13, chap. F1.2].) In particular, we find that for each $\vec{x} \in U$ and $\vec{d} \in D(n)$, $\vec{x} + \vec{d}$ lies in U again. We recall that $\vec{x}$ and $\vec{y}$ are i-neighbours for the nil-square i-structure if and only if $\vec{x} - \vec{y}$ lies in $D(n)$. This shows that a formally open subobject of R^n is *i-closed* (see definition 3.2.7 below), i.e. $\mathcal{S}$ satisfies the sequent

$$\partial_n[\iota(x), \vec{y}/\vec{x}, \vec{y}] \ \vdash_{x:U,\vec{y}} \ (\exists! y : U)\iota(y) = \vec{y}$$

In less formal terms: if $\vec{x} \in U$ and $\vec{x} \sim_1 \vec{y}$ in R^n, then $\vec{y} \in U$. From the i-closure of $\iota : U \rightarrowtail R^n$ we can easily deduce that U together with the nil-square i-structure (the pullback of the nil-square i-structure along ι) is closed under i-affine combinations and hence an i-affine subspace of R^n. Moreover, the monomorphism $\iota : U \to R^n$ reflects i-structure.

(b) For a well-adapted model $(\mathcal{S}, \iota)$ and a smooth manifold M an open inclusion $U \rightarrowtail M$ is mapped to a formally open subobject $\iota(U) \rightarrowtail \iota(M)$ ([Joh13, lem. F1.3.4(i)], or [Koc06, thm. III.3.4]). Since ι maps open covers $U_j \rightarrowtail M$ to jointly epimorphic families $\iota(U_j) \rightarrowtail \iota(M)$, $j \in I$, and since $\bigvee_{j \in I} U_j$ is the image- factorisation of the unique morphism $p : \coprod_{j \in I} \iota(U_j) \to M$ induced by the monomorphisms $U_j \rightarrowtail M$, we find $\bigvee_{j \in I} U_j = M$.

Definition 3.2.7. Let U and A be i-structures in a finite-limit category C, and $j : U \rightarrowtail A$ an i-morphism. We say that U is **i-closed**, if C satisfies the following sequent

$$A\langle 2\rangle(j(x), y) \ \vdash_{x:U,y:A} \ (\exists! z : U) j(z) = y \wedge U\langle 2\rangle(x, z)$$

Theorem 3.2.8. *Let $\mathcal{S}$ be a Grothendieck topos and R a non-trivial K-algebra of line type. Let* $\mathrm{Mfd}(\mathcal{S})$ *denote the full subcategory of formal manifolds in $\mathcal{S}$. Every formal n-dimensional manifold M carries a unique i-affine structure such that for any chart (ι, ϕ), $\iota : U \rightarrowtail M$ is an i-structure reflecting i-affine map and U is i-closed. There is a functor $Af : \mathrm{Mfd}(\mathcal{S}) \to \mathrm{IAff}(\mathcal{S})$ mapping each M to this i-affine space such that $U \circ Af = I_{\mathrm{Mfd}(\mathcal{S})}$, where U denotes the forgetful functor $U : \mathrm{IAff}(\mathcal{S}) \to \mathcal{S}$.*

Proof. (1) Let M be a formal n-dimensional manifold. We consider the (maximal) atlas $\iota_j : U_j \rightarrowtail M$, $j \in I$ consisting of all n-dimensional charts (ι_j, ϕ_j) for M. Let J denote the small preorder in the slice category $\mathcal{S}/M$ generated by this atlas. Formally open subobjects are stable under pullback; in particular, a finite intersection of formally open subobjects is formally open again. The intersection of two n-dimensional charts (as subobjects of M) yields an n-dimensional chart again. In particular, J has finite meets. The union $\bigvee_{j \in I} U_j$ is thus the colimit of the inclusion $J \rightarrowtail \mathcal{S}$.

Each chart $\phi_j : U_j \rightarrowtail R^n$ is an i-structure reflecting, i-closed, i-affine subspace of R^n. Because of that, and since $U_k \rightarrowtail U_j$ composed with ϕ_j yields a chart, every morphism $U_k \rightarrowtail U_j$ is an i-structure reflecting, i-closed, i-affine subspace. The diagram $J \rightarrowtail \mathcal{S}$ thus yields a diagram $J \rightarrowtail \mathrm{IAff}(\mathcal{S})$ of i-structure reflecting i-affine maps. By theorem 2.6.19 there is a unique i-affine structure on M making it the colimit of $J \rightarrowtail \mathrm{IAff}(\mathcal{S})$ in $\mathrm{IAff}(\mathcal{S})$.

(2) The monomorphisms $\iota_j : U_j \rightarrowtail M$ are the components of the colimit cone of $J \rightarrowtail \mathrm{IAff}(\mathcal{S})$. They thus reflect i-structure by theorem 2.6.19. We need to show the U_j i-closed subspaces of M. We shall give a naive proof only.

Suppose $M\langle 2\rangle(j(x), y)$ hold for an $x \in U_j$. By the construction of the i-structure on $\bigvee_{j \in I} U_j$ as the union of the images of the i-structures under the monomorphisms $j : U_j \to M$, there is a $k : U_k \rightarrowtail M$ and $x_0, y_0 \in U_k$ such that $U_k\langle 2\rangle(x_0, y_0)$ and $k(x_0) = j(x)$ and $k(y_0) = y$. Consider the chart, which is the intersection of U_j and U_k

$$\begin{tikzcd} U_j \cap U_k \arrow[r, tail, "k_2"] \arrow[d, tail, "k_1"] & U_j \arrow[d, tail, "j"] \\ U_k \arrow[r, tail, "k"] & M \end{tikzcd}$$

There is a $z_0 \in U_j \cap U_k$ such that $k_1(z_0) = x_0$ and $k_2(z_0) = x$. Since $U_j \cap U_k$ is an i-closed subspace of U_k there is a $z_1 \in U_j \cap U_k$ such that $(U_j \cap U_k)\langle 2\rangle(z_1, z_0)$ and $k_1(z_1) = y_0$; but then we find $U_j\langle 2\rangle(x, k_2(z_1))$ and $jk_2(z_1) = y$. This shows that U_k is i-closed.

Clearly, any i-affine structure on M, for which all (n-dimensional) charts $U \rightarrowtail M$ are i-structure reflecting and i-closed, has to coincide with the one constructed in (1). Indeed, since the charts are covering, i-closed and reflect i-structure, the i-structure has to be the join of the images of the i-structures on the charts; which are themselves uniquely determined by the nil-square i-structure on R^n. Since the charts are i-closed, i-affine combinations can be computed in the charts (and hence in R^n) as well. Furthermore, the i-affine structure on M can be obtained from any atlas for M.

(3) Let $f : M \to N$ be an $\mathcal{S}$-morphism between two formal manifolds. We wish to show f an i-affine map. Once again, we will give a naive proof only. Firstly, we assume that $M \rightarrowtail R^n$ and $N \rightarrowtail R^m$ are formally open. Due to the Kock-Lawvere axiom for $D(n)$ and since M is i-closed we can write $f : M \to R^m$ for two i-neighbours $x, y \in U$ as $f(y) - f(x) = \partial f(x)[y - x]$. Since $y - x \in D(n)$, and since $\partial f : R^n \to R^m$ is R-linear, we have $f(x) - f(y) \in D(m)$. This shows f an i-morphism. A calculation as in part (2) of the proof of theorem 3.1.5 shows that f preserves i-affine combinations. Since $N \rightarrowtail R^m$ reflects i-structure we may conclude that $f : M \to N$ is i-affine.

We consider general formal manifolds M and N. $f : M \to N$ is i-affine if and only if we find an atlas $U_j \rightarrowtail M$, $j \in I$ for M, such that the restriction of f to every chart $U_j \rightarrowtail M$ is i-affine. Let $x \in M$ and let $V \rightarrowtail N$ be a chart that contains $f(x)$. Since formally open subobjects are stable under pullback $f^{-1}(V)$ is a formally open subobject of M containing x. Let $U \rightarrowtail M$ be a chart containing x, then $f^{-1}(V) \cap U_j \rightarrowtail M$ is also a chart containing x. Moreover, if we restrict $f : M \to N$ to the chart $f^{-1}(V) \cap U_j$, then this restriction factors through V

$$\begin{tikzcd} f^{-1}(V) \cap U_j \arrow[rr, "f"] \arrow[dr] & & N \\ & V \arrow[ur, tail] & \end{tikzcd}$$

From an atlas $V_j \rightarrowtail M$, $j \in I$ and an atlas for M we can construct an atlas $U_j \rightarrowtail M$, $j \in I$ such that the restriction of f to U_j factors through V_j. Since such morphisms have been shown i-affine, the morphism f must be i-affine.

We obtain a functor $Af : \mathrm{Mfd}(\mathcal{S}) \to \mathrm{IAff}(\mathcal{S})$ as asserted.

□

Remark 3.2.9. For the proof of theorem 3.2.8 we only required that the the charts are *1-open* subobjects, i.e. the pullback diagram in definition 3.2.5(1) only holds for $D_W = D$.

Remark 3.2.10 (*Schemes*)**.** A parallel result should be true for schemes in algebraic geometry, if we adopt the functor-of-points approach to schemes [DG80]. We sketch the idea for a proof.

A *scheme* is defined as a sheaf M on the (large) site $(\mathbb{Z}\text{-Alg}, Z)$, where Z is the *Zariski-coverage*, such that there is a family $U_j \rightarrowtail M$, $j \in I$, of *open*[2] As for formal manifolds we can consider the preorder J of affine open subschemes of M. Unlike for manifolds the intersection of two open affine subschemes is not necessarily an open affine subscheme of M (unless M is semi-separated). M is the colimit of the diagram induced by J. We know that every

[2]We shall omit the definition of what 'open' means here. representable subobjects and $M = \bigvee_{j \in I} U_j$.

representable $y(A)$ is an i-affine space over its nil-square i-structure. (Cf. theorem 2.3.5) We need to show that the inclusions of open affine schemes reflect i-structure. Then we can apply theorem 2.6.19 to give M a unique i-affine structure glued together from the i-affine structures of the representables. Since the (meta)category of sheaves on the large site ($\mathbb{Z}$-Alg$, Z)$ is not a topos, we have to check carefully whether the proof of theorem 2.6.19 still works in this case. As regards showing that every morphism of schemes $f : M \to N$ is i-affine, since open subschemes are stable under pullback, we can reduce it to the case of morphisms between open affine subschemes; but we know that these are i-affine already.

If we pass from the large Zariski site ($\mathbb{Z}$-Alg$, Z$) to the small Zariski site ($\mathbb{Z}$-Alg$_{fp}, Z$) then we do have a topos Sh($\mathbb{Z}$-Alg$_{fp}, Z$). Repeating the definition of schemes in this topos yields the schemes that are *locally of finite presentation* [Low16, example 3.5.22(i)]. In this case we only need to check that inclusions between open affine subschemes reflect i-structure.

Another important class of spaces in SDG are *infinitesimally smooth spaces*, which are defined as follows (cf. [Joh13, def. F1.2.5]). Let $\mathcal{S}$ be a Grothendieck topos with a K-algebra object R of line type, where K is a field. The map that assigns a space D_W to the Weil algebra W induces a functor $D_{(-)} : \mathrm{Weil}(K)^{op} \to \mathcal{S}$, where Weil($K$) denotes the full subcategory of Weil algebras $\mathrm{Weil}(K) \rightarrowtail K\text{-Alg}_\omega$. $D_{(-)}$ is the restriction of the finite-limit preserving functor $F_R : K\text{-Alg}_\omega^{op} \to \mathcal{S}$, which is induced by the K-algebra object R. The subcategory Weil(K) is closed under finite connected limits [Joh13, lem. F1.1.11]. We say that an object M in $\mathcal{S}$ is **infinitesimally smooth** (*i-smooth*) if the functor $[D_{(-)}, M] : \mathrm{Weil}(K) \to \mathcal{S}$ preserves finite connected limits.

The full subcategory of i-smooth spaces is an exponential ideal of $\mathcal{S}$ (i.e. for every i-smooth M and any X, $[X, M]$ is i-smooth), and it is closed under limits. Since R is of line type, R is i-smooth; hence R^n is i-smooth. Formally open subobjects of R^n are i-smooth and with this one can show that a formal manifold is i-smooth as well [Joh13, lem. F1.2.11].

Since every $F_R(A)$ can be considered as a zero-locus of finitely many morphisms $R^n \to R$, it is, in particular, an equaliser of a finite family of morphisms $R^n \to R$, and hence i-smooth. The other example of i-affine spaces we have discussed are formal manifolds, which are i-smooth as well.

Example 3.2.11. Let R be non-trivial. An example of an i-affine space that is not i-smooth is given by the *infinitesimal cross* A, which is the pushout

$$\begin{tikzcd} 1 \arrow[r, tail, "0"] \arrow[d, tail, "0"'] & D \arrow[d, tail, "j"] \\ D \arrow[r, tail, "k"] & A \end{tikzcd}$$

in $\mathcal{S}$. (Recall that in $\mathcal{S}$ a pushout of a monic is a monic, and the pushout square is also a pullback square [MM92, cor. IV.10.4].) Since 1 is a total i-affine subspace of D, by corollary 2.6.16, there is a unique i-affine structure on A that makes the above diagram a pushout in $\mathrm{IAff}(\mathcal{S})$. Suppose the object A is i-smooth. Consider the diagram

$$\begin{tikzcd} 1 \arrow[r, tail, "0"] \arrow[d, tail, "0"'] & D \arrow[d, tail, "i_2"] \\ D \arrow[r, tail, "i_1"] & D(2) \end{tikzcd}$$

This square is the $D_{(-)}$ image of a pullback diagram in $\mathrm{Weil}(K)$. Since A is a pushout, there is a unique morphism $\ell : A \longrightarrow D(2)$ such that $\ell k = i_1$ and $\ell j = i_2$. If A were i-smooth, then there would be a unique morphism $h : D(2) \longrightarrow A$, such that $hi_1 = k$ and $hi_2 = j$. This yields $h\ell = 1_A$. Since $D(2)$ is i-smooth, we also find $\ell h = 1_{D(2)}$ and hence $D(2) \cong A$.
The space $D(2)$ is thus the join of the subobjects i_1 and i_2 of $D(2)$. Using a syntactical description of these subobjects yields

$$D(2) = \{(d_1, d_2) \in D(2) \mid (d_1 = 0) \vee (d_2 = 0)\}$$

The diagonal $\Delta : D \rightarrowtail D \times D$ factors through $D(2)$. The element $\Delta(d) = (d, d)$ lies in the union of i_1 and i_2 if and only if $d = 0$; but since the union is $D(2)$ this means that the diagonal Δ has to factor through 1. This implies $D \cong 1$, and hence $0 = 1$ in R, for R, as a ring of line type, satisfies the cancellation rule

$$(\forall d : D) dx = 0 \vdash_{x:R} x = 0$$

But that contradicts our assumption of R being non-trivial. We conclude that A cannot be i-smotth.

3.3 Naive Synthetic Differential Geometry

In this section we develop some basic i-algebra of loci and formal 1-manifolds in the context of naive SDG. The presentation will be along the lines of [Koc09], but we shall employ a slightly different terminology. Firstly, we introduce formal 1-manifolds and loci, show that they are i-affine spaces, and prove that all maps between them are i-affine. Secondly, we study the tangent bundle over an i-affine space with the aim to develop the R-linear structure on each tangent space from the i-affine structure of the space step by step.

3.3.1 Formal 1-manifolds and loci

Let R be a $\mathbb{Q}$-algebra that satisfies the *Kock-Lawvere axioms* for each space

$$D(n) = \{(d_1, .., d_n) \in R^n \mid d_i d_j = 0,\ 1 \leq i, j \leq n\}$$

and $n \geq 1$; that is for every map $f : D(n) \to R$ there are unique $a_0, \dots, a_n \in R$ such that for all $(d_1, \dots, d_n) \in D(n)$

$$f(d_1, \dots, d_n) = a_0 + \sum_{i=1}^{n} a_i d_i \tag{K-L}$$

The space $D(n)$ can be characterised as the common radical of all quadratic forms $q : R^n \to R$, where q is called quadratic if there is an R-bilinear form $\varphi : R^n \times R^n \to R$ such that $q(x) = \varphi(x, x)$. A further consequence of (K-L) is that every map $f : R^m \to R^n$ is differentiable at each $P \in R^n$; that is for every P there is a unique R-linear map $\partial f(P)$ such that $f(P + d) - f(P) = \partial f(P)[d]$ for each $d \in D(n)$.

We define a relation '$\sim$' on R^n:

$$P \sim Q \quad \Longleftrightarrow \quad P - Q \in D(n)$$

and say that P and Q are *i-neighbours*, if $P \sim Q$. The relation '$\sim$' is clearly symmetric and reflexive, and hence generates an i-structure

$$R^n\langle m\rangle = \{(P_1, \dots, P_m) \in (R^n)^m \mid P_i \sim P_j,\ 1 \leq i, j \leq m\}, \qquad m \in \mathbb{N}$$

We write $\langle P_1, \dots, P_m\rangle$, if $(P_1, \dots, P_m) \in R^n\langle m\rangle$ and call such an m-tuple an **i-neighbourhood** in R^n. We set $\mathcal{A}(n) = \{(\lambda_1, \dots, \lambda_n) \mid \sum_{i=1}^{n} \lambda_i = 1\}$.

Lemma 3.3.1.

(1) *Let $\lambda^k \in \mathcal{A}(n)$, $1 \leq k \leq m$, and let $\langle P_1, \dots, P_n \rangle$ be an i-neighbourhood in R^n, then*

$$\Big(\sum_{j=1}^{n} \lambda_j^1 P_j, \dots, \sum_{j=1}^{n} \lambda_j^m P_j\Big) \in R^n\langle m\rangle$$

In particular, R^n is an i-affine space.

(2) *Every $f : R^n \to R^m$ is an i-affine map.*

Proof. (1) This follows from the same calculation that we gave in part (1) of the proof of theorem 2.3.1.

(2) Let $P \sim Q$. We have $f(P) - f(Q) = \partial f(Q)[P - Q]$. Since $D(n)$ is stable under R-linear maps, we obtain $f(P) \sim f(Q)$. The calculation that we gave in part (2) of the proof of theorem 3.1.5 shows that f preserves affine combinations of i-neighbourhoods.

□

We say that a subspace $U \rightarrowtail R^n$ is *i-closed*, if with $P \in U$ and $d \in D(n)$, $P + d$ lies also in U, and hence $P \in U$ and $P \sim Q$ implies $Q \in U$. In particular, for every i-neighbourhood $(P_1, \dots, P_m) \in R^n\langle m\rangle$ of points in U their affine combinations have to lie in U as well. This shows U an i-affine subspace of R^n, if we equip U with the i-structure $U\langle m\rangle = R^n\langle m\rangle \cap U^m$.

Let M be a space and $U \rightarrowtail M$ a subspace. We say that U is *formally* 1-*open*, if for all $n \geq 1$ and for every map $t : D(n) \to M$ it holds that t factors through $U \rightarrowtail M$ if and only if $t(0) \in U$. We call M a **formal 1-manifold** of dimension n, if there is a covering family of formally 1-open subspaces $U_j \rightarrowtail M$, $j \in I$, such that there is a $\phi_j : U_j \rightarrowtail R^n$, which is an i-closed subspace of R^n. A formally 1-open subspace U of M

$$\begin{tikzcd} & U \arrow[dl, tail, "\iota"'] \arrow[dr, tail, "\phi"] & \\ M & & R^n \end{tikzcd}$$

that is also an i-closed subspace of R^n is called a *chart* (of dimension n) and a covering family of charts of dimension n is called an (n-dimensional) *atlas* for M.

Proposition 3.3.2.

(1) *Let M be a formal 1-manifold, then M is an i-affine space.*

(2) *Every map $f : M \to N$ between formal manifolds is i-affine.*

Proof. (1) We consider the maximal atlas of charts $\phi_j : U_j \rightarrowtail R^n$ with $\iota_j : U_j \rightarrowtail M$, $j \in I$, of the same dimension n as the manifold M, and define the i-structure on M by taking $M\langle n\rangle = \bigvee_{j\in I} \iota_j^n(U_j\langle n\rangle)$ to be the join of the images $\iota_j^n(U_j\langle n\rangle)$. Since M is covered by the atlas this defines an i-structure on M. Indeed, let $\langle P_1, \ldots, P_n\rangle$ be an i-neighbourhood in M and $\lambda \in \mathcal{A}(n)$. By definition of the i-structure there is a chart $\iota_j : U_j \rightarrowtail M$ and an i-neighbourhood $\langle Q_1, \ldots, Q_n\rangle$ in U_j such that $\iota_j(Q_k) = P_k$. Define

$$\sum_{k=1}^{n} \lambda_k P_k = \iota_j\Big(\sum_{k=1}^{n} \lambda_k Q_k\Big)$$

We need to show that this is independent of the choice of ι_j. Suppose there is another chart $\iota_\ell : U_\ell \rightarrowtail M$ and $\langle X_1, \ldots, X_n\rangle$ in U_ℓ such that $\iota_\ell(X_k) = P_k$. We form the intersection $U_j \cap U_\ell \rightarrowtail M$ of subspaces of M. $U_\ell \cap U_j$ is clearly formally 1-open. From this, and because U_j is i-closed, we can conclude that $\phi_j|_{U_j\cap U_\ell} : U_j \cap U_\ell \rightarrowtail U_j \rightarrowtail R^n$ is i-closed as well. In particular, $U_j \cap U_\ell$ is an i-closed i-affine subspace of U_j and U_ℓ. An argument like in part (2) of the proof of theorem 3.2.8 then shows that $\iota_j(\sum_{k=1}^n \lambda_k Q_k) = \iota_\ell(\sum_{k=1}^n \lambda_k X_k)$. The axioms of an i-affine space then follow, since they hold in each chart.

(2) See part (3) of the proof of theorem 3.2.8.

□

Remark 3.3.3. Let $U \rightarrowtail R^n$. Due to (K-L) U is i-closed if and only if $U \rightarrowtail R^n$ is formally 1-open. We could have thus given the definition of a formal 1-manifold M just in terms of formally 1-open subobjects of M, which are simultaneously also formally 1-open subobjects of R^n.

Another possible approach would have been to not restrict ourselves to formally 1-open subobjects only, and to define a formal 1-manifold as a structure having an n-dimensional atlas; that is a covering family of subobjects $U_j \rightarrowtail M$, $j \in I$, such that $\phi_j : U_j \rightarrowtail R^n$ is i-closed and for every $j, \ell \in I$ the subobjects $\phi_j|_{U_j\cap U_\ell} : U_j \cap U_\ell \to R^n$ and $\phi_\ell|_{U_j\cap U_\ell} : U_j \cap U_\ell \to R^n$ are i-closed, too.

This definition is more like the definition of a manifold in classic differential geometry. Since it does not restrict to charts on formally 1-open subobjects it is also more general than our original definition. However, the advantage of the original definition is that being a formal 1-manifold is a property of a space rather than an extra structure.

The second type of geometric space we would like to introduce here are loci. Let M be a formal 1-manifold. A subspace $L \rightarrowtail M$ is called a **locus**, if there is a family of maps $f_j : M \to R$, $j \in I$, such that $L = \bigcap_{j\in I} f_j^{-1}(0)$, and L satisfies the **Tietze axiom**:

For each map $f : L \to R$ and each point P there is a chart $U \rightarrowtail M$ containing P and a map $\bar{f} : U \to R$ extending the map $f : L \cap U \to R$.

Proposition 3.3.4.

(1) *Let $L \rightarrowtail M$ be a locus, then L is an i-affine subspace of M that reflects i-structure.*

(2) *Let N a formal 1-manifold. Every map $f : L \to N$ is i-affine, and every map $g : N \to L$ is i-affine.*

(3) *Every map $f : L \to L'$ between loci is i-affine.*

Proof. (1) Define the i-structure on L by $L\langle n\rangle = M\langle n\rangle \cap L^n$. We show that L is closed under i-affine combinations. Consider $\langle P_1, \dots, P_n\rangle$ in L and $\lambda \in \mathcal{A}(n)$. Let $f_j : M \to R$, $j \in I$, be a family of functions of which L is the common zero-locus. Since each f_j is i-affine, we find

$$f_j\Big(\sum_{k=1}^{n} \lambda_k P_k\Big) = \sum_{k=1}^{n} \lambda_k f_j(P_k) = 0$$

and hence $\sum_{k=1}^{n} \lambda_k P_k \in L$. This shows L an i-affine space and the inclusion $L \rightarrowtail M$ an i-affine, i-structure reflecting map.

(2) The map $g : N \to L \rightarrowtail M$ is i-affine. Since the inclusion $L \rightarrowtail M$ is an i-structure reflecting injective map, the map $g : N \to L$ must be i-affine. By the the Tietze axiom we find for each $P \in L$ a chart $U \rightarrowtail M$ containing P and a map $\bar{f} : U \to N$, which is an extension of $f|U : U \cap L \to N$. We know that $\bar{f}$ is i-affine, hence so is $f|U$. Since this is true at any point $P \in N$, f must be i-affine.

(3) This follows from (2) and since $L' \rightarrowtail N'$ reflects i-structure.

□

Let $A\langle -\rangle$ be an i-structure on a space A, $U_j \rightarrowtail A$, $j \in I$ a covering family and let $r_j : U_j \to R$, $j \in I$ be a family of maps. We say that the i-structure on A is **locally initial** with respect to the family of maps r_j, if it holds true that for every $n \geq 2$ and every $(P_1, \dots, P_n) \in A^n$

$$(P_1, \dots, P_n) \in A\langle n\rangle \iff (\forall j \in I|(P_1, \dots, P_n))\, (r_j(P_1), \dots, r_j(P_n)) \in R\langle n\rangle,$$

where $I|(P_1, \dots, P_n)$ is the subset of I containing all the indices $j \in I$ such that $(P_1, \dots, P_n) \in U_j^n$.

Proposition 3.3.5.

(1) *Let M be a formal 1-manifold, then the i-structure on M is locally initial with respect to all maps $U \to R$ defined on domains of charts $U \rightarrowtail M$.*

(2) *Let $L \rightarrowtail M$ be a locus, then the i-structure on L is locally initial with respect to all maps $U \cap L \to R$ defined on domains of charts $U \rightarrowtail M$.*

Proof. (1) Let $U \rightarrowtail M$ be a chart. We know that every map $U \to R$ is i-affine, and, in particular, an i-morphism. This shows one direction. As regards the other direction, it is sufficient to show that the i-structure on R^n is initial for all maps $R^n \to R$. Indeed, assume the latter, consider an i-closed subspace $U \rightarrowtail R^n$ and let $f(P) \sim f(Q)$ for all maps $f : U \to R$. In particular, this holds true for all maps $f : R^n \to R$, so $P \sim Q$ in R^n; but $U \rightarrowtail R^n$ reflects i-structure, as U is i-closed, so $(P, Q) \in U\langle 2\rangle$, and hence in $M\langle 2\rangle$.
We consider $M = R^n$. It is sufficient to show that $(r(P) - r(Q))^2 = 0$ for every map $r : R^n \to R$ implies $P - Q \in D(n)$. Indeed, $r = \mathrm{pr}_j$ yields $(P_i - Q_i)^2 = 0$, and $r = \mathrm{pr}_i + \mathrm{pr}_j$ yields $0 = ((P_i - Q_i) + (P_j - Q_j))^2 = 2(P_i - Q_i)(P_j - Q_j)$. Since 2 is invertible in R we obtain $(P_i - Q_i)(P_j - Q_j) = 0$ for all $1 \leq i, j \leq n$, and hence $P - Q \in D(n)$ as asserted.

(2) This follows from (1), the Tietze axiom, and since the inclusion $L \rightarrowtail M$ reflects i-structure.

□

Remark 3.3.6. If the i-structure on A is locally initial with respect to some family of maps, then the i-structure on A is generated by $A\langle 2\rangle$.

3.3.2 The tangent bundle of an infinitesimally affine space

Let M be an i-affine space and X any space. We define an i-affine structure on the function space M^X in a pointwise manner:

- $M^X\langle n\rangle = \{(f_1, \ldots, f_n) \in (M^X)^n \mid (\forall x \in X)\ \langle f_1(x), \ldots, f_n(x)\rangle \in A\langle n\rangle\}$
- For an i-neighbourhood $\langle f_1, \ldots, f_n\rangle$ and $\lambda \in \mathcal{A}(n)$ we define a map $\sum_{k=1}^n \lambda_k f : X \to A$ by $(\sum_{k=1}^n \lambda_k f)(x) = \sum_{k=1}^n \lambda_k f_k(x)$.

As regards maps between function spaces we observe:

Proposition 3.3.7.

(1) *Every i-affine map $f : M \to N$ induces an i-affine map $f^X : M^X \to N^X$.*

(2) *For each $P \in X$ the evaluation-at-P map $\mathrm{ev}_P : M^X \to M$ is i-affine.*

(3) *Let N be an i-affine space and $f : N \to M^X$ a map, such that $\mathrm{ev}_P \circ f : N \to M$ is i-affine for each $P \in X$. The map f is i-affine if and only if it preserves i-structure.*

Proof. (1) and (2) follow from the pointwise definition of the i-structure on function spaces immediately. As for (3), if f is i-affine then it clearly preserves i-structure. Conversely, let f preserve i-structure. We have to show that it preserves i-affine combinations. Consider an i-neighbourhood $\langle Q_1, \dots, Q_n \rangle$ in N and a $\lambda \in \mathcal{A}(n)$. Since f preserves i-structure $\langle f(Q_1), \dots, f(Q_n) \rangle$ is an i-neighbourhood in M^X, and, since each $\mathrm{ev}_P \circ f$ is i-affine, we find

$$f\Big(\sum_{k=1}^{n} \lambda_k Q_k\Big)(P) = \sum_{k=1}^{n} \lambda_k f(Q_k)(P)$$

But this is just saying that f preserves i-affine combinations. □

Let M be an i-affine space, and $P \in M$. The **i-neighbourhood** $D(P)$ **of** P is defined by

$$D(P) = \{Q \in M \mid \langle P, Q \rangle \in M\langle 2 \rangle\}$$

We can define an i-structure on $D(P)$ as follows:

$$D(P)\langle n \rangle = \{(P_1, \dots, P_n) \in D(P)^n \mid \langle P, P_1, \dots, P_n \rangle \in M\langle n+1 \rangle\}$$

Proposition 3.3.8. *Let M be an i-affine space and $P \in M$. The i-neighbourhood $D(P)$ of P is an i-vector space* (*over R*).

Proof. Let $\lambda \in R^n$ and let $\langle P_1, \dots, P_n \rangle$ an i-neighbourhood in $D(P)$. We define $\sum_{k=1}^{n} \lambda_k P_k$ by $(1 - \sum_{k=1}^{n} \lambda_k)P + \sum_{k=1}^{n} \lambda_k P_k$. Note that the latter is an i-affine combination, which is well-defined, for $\langle P, P_1, \dots, P_n \rangle$ is an i-neighbourhood in M by the definition of the i-structure on $D(P)$. It is straightforward to check that this makes $D(P)$ an i-algebra of the clone of R-linear combinations. □

Remark 3.3.9. The space $D(n)$ is the i-neighbourhood $D(0)$ of 0 in R^n, and thus an i-vector space. Note that R^n is not an i-vector space over its nil-square i-structure, as not every point in R^n is an i-neighbour of 0.

With this we are ready to study the tangent bundle $\mathrm{ev}_0 : M^D \to M$ of an i-affine space M, where $D = D(1) = \{d \in R \mid d^2 = 0\}$. From the preceding propositions we can make the following general observation:

Corollary 3.3.10.

(1) M^D *is an i-affine space, and* ev_0 *is an i-affine map.*

(2) *Fix* $x \in M$ *and consider the constant function* c_x *with value* x*. The i-neighbourhood* $D(c_x)$ *of* c_x *is an i-vector space. Moreover, since* $D(c_x) = D(x)^D$ *we can see that the i-R-linear structure on* $D(c_x)$ *is the pointwise i-R-linear structure on* $D(x)^D$*.*

The *tangent space* T_xM at $x \in M$ is defined as $\mathrm{ev}_0^{-1}(0)$. We define the **reduced tangent space** $\hat{T}_xM$ at x by $T_xM \cap D(x)^D$. It becomes an i-vector subspace of $D(x)^D$ if we equip it with the i-structure $\hat{T}_xM\langle n\rangle = (\hat{T}_xM)^n \cap D(x)^D\langle n\rangle$. In particular, the i-$R$-linear structure is defined pointwise.

Note that this is a completely general result that does not rely on any form of Kock-Lawvere axiom whatsoever. The next proposition shows what happens, if we make further assumptions about the tangent vectors $t : D \to M$, or M.

Proposition 3.3.11.

(1) *If each map* $t : D \to M$ *is an i-morphism, then* $T_xM = \hat{T}_xM$ *is an i-vector subspace of* $D(x)^D$ *for each* $x \in M$*.*

(2) *If the i-structure on* M *is locally initial with respect to a family of maps* $r_j : U_j \to R$*,* $j \in I$*, and the* $U_j \rightarrowtail M$ *are formally* 1*-open, then the i-R-linear structure on* T_xM *is total, i.e.,* T_xM *is an R-vector space.*

Proof. (1) If $t : D \to M$ is an i-morphism, then $\langle t(d), t(0)\rangle$ in M, since 0 is an i-neighbour of each $d \in D$. But then we have $t \in D(t(0))^D$, and hence $T_xM \subset D(x)^D$ for $x = t(0)$.

(2) Let $j \in I|t(0)$. Since U_j is formally 1-open t factors through U_j. Every map $r_jt : D \to R$, $j \in I|t(0)$ is i-affine due to (K-L), and hence an i-morphism. Since the i-structure on M is locally initial with respect to the maps r_j, this shows each $t : D \to M$ an i-morphism. By (1) T_xM is an i-vector subspace of $D(x)^D$. Let $t_1, t_2 : D \to M$ be two tangent vectors at $x \in M$. We consider the map $(r_jt_1, r_jt_2) : D \to R^2$ for $j \in I|x$. By (K-L) this map is i-affine and thus an i-morphism as well. In particular, we have $(r_j \circ t_1(d) - r_j(x), r_j \circ t_2(d) - r_j(x)) \in D(2)$ which yields $(r_j \circ t_1(d) - r_j \circ t_2(d))^2 = 0$. Since this is true for all $j \in I|(t_1(d), t_2(d))$, we have $(t_1(d), t_2(d)) \in M\langle 2\rangle$ for all $d \in D$; but this implies that t_1 and t_2 are i-neighbours in $D(x)^D$. The i-structure on M is generated by $M\langle 2\rangle$, hence so is the i-structure on $D(x)^D$ and T_xM. The i-structure on T_xM is thus total.

□

Remark 3.3.12. Since the i-structures on loci and formal 1-manifolds are locally initial with respect to a family of R-valued maps defined on domains of charts, we obtain that their tangent spaces are all R-vector spaces. We also see that the R-linear structure on the tangent space T_xM is induced by the i-R-linear structure on the i-neighbourhood $D(x)$, which is induced by the i-affine structure on the space itself.

In [Koc09] the R-linear structure on T_xM is also obtained by pointwise constructions using i-affine combinations in M. In this respect our result is not new. What is new, is the conceptual and structural clarification of this construction. However, what is still missing is a characterisation of i-affine spaces M, for which the i-R-linear structure on $D(x)^D$ induces a total R-linear structure on the tangent spaces T_xM for each $x \in M$.

Remark 3.3.13. Note that D is a locus due to (K-L). If M is a manifold or a locus every tangent vector $t : D \to M$ must thus be i-affine; in particular, it is an i-R-linear map $D \to D(t(0))$.

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